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Geometry of all supersymmetric type I backgrounds

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Abstract: We find the geometry of all supersymmetric type I backgrounds by solving the gravitino and dilatino Killing spinor equations, using the spinorial geometry technique, in all cases. The solutions of the gravitino Killing spinor equation are characterized by their isotropy group in $\mathrm{Spin}(9,1)$, while the solutions of the dilatino Killing spinor equation are characterized by their isotropy group in the subgroup $\Sigma(\mathcal{P})$ of $\mathrm{Spin}(9,1)$ which preserves the space of parallel spinors $\mathcal{P}$. Given a solution of the gravitino Killing spinor equation with L parallel spinors, $L = 1, 2, 3, 4, 5, 6, 8$, the dilatino Killing spinor equation allows for solutions with N supersymmetries for any $0 < N \leq L$. Moreover for $L = 16$, we confirm that $N = 8, 10, 12, 14, 16$. We find that in most cases the Bianchi identities and the field equations of type I backgrounds imply a further reduction of the holonomy of the supercovariant connection. In addition, we show that in some cases if the holonomy group of the supercovariant connection is precisely the isotropy group of the parallel spinors, then all parallel spinors are Killing and so there are no backgrounds with $N < L$ supersymmetries.

Keywords: Superstring Vacua, Superstrings and Heterotic Strings, Flux compactifications, Supergravity Models.

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Contents

1. Introduction

The last twenty years, supersymmetric solutions of the type I supergravities and their geometries have been the focus of intensive investigation because of their applications in type I and heterotic superstrings, see e.g. [1]–[21]. Type I supergravities have three types of Killing spinor equations associated with the vanishing of the supersymmetry variations of the gravitino, dilatino and gaugino. The gravitino Killing spinor equation is a parallel transport equation of a metric connection with skew-symmetric torsion, $\hat{\nabla}$, where the torsion is the NS⊗NS or R⊗R three-form field strength in the heterotic[1] or type I superstrings, respectively. So the holonomy of $\hat{\nabla}$, $\mathrm{hol}(\hat{\nabla})$, is contained in $\mathrm{Spin}(9,1)$. The existence of parallel spinors requires that $\mathrm{hol}(\hat{\nabla})$ must be a subgroup of their isotropy group in $\mathrm{Spin}(9,1)$. Therefore either the Killing spinors have a non-trivial (proper) stability Lie subgroup in $\mathrm{Spin}(9,1)$ or the stability subgroup is $\{1\}$ and the curvature $\hat{R}$ of $\hat{\nabla}$ vanishes, $\hat{R}=0$. The isotropy or stability subgroups, up to a discrete identification, of Majorana-Weyl spinors in $\mathrm{Spin}(9,1)$ are

$$
\begin{aligned}
&\mathrm{Spin}(7)\ltimes\mathbb{R}^8\ (1)\supset\mathrm{SU}(4)\ltimes\mathbb{R}^8\ (2)\supset\mathrm{Sp}(2)\ltimes\mathbb{R}^8\ (3)\supset(\mathrm{SU}(2)\times\mathrm{SU}(2))\ltimes\mathbb{R}^8\ (4)\\
&\qquad\qquad\supset\mathrm{SU}(2)\ltimes\mathbb{R}^8\ (5)\supset\mathrm{U}(1)\ltimes\mathbb{R}^8\ (6)\supset\mathbb{R}^8\ (8)\,,\\
&\mathrm{Spin}(7)\ltimes\mathbb{R}^8\ (1)\supset G_2\ (2)\supset\mathrm{SU}(3)\ (4)\supset\mathrm{SU}(2)\ (8)\supset\{1\}\ (16)\,,
\end{aligned}
\tag{1.1}
$$

where in parenthesis we have denoted the number of linearly independent invariant spinors. The maximal compact subgroups of (1.1) have appeared before, see [26], in the context of supersymmetric M-brane configurations. Lists of isotropy groups of $\mathrm{Spin}(9,1)$ and $\mathrm{Spin}(10,1)$ spinors in various representations[2] can be found in [27]. Most of above groups have also appeared in [28]. To our knowledge the concise list of isotropy groups of $\mathrm{Spin}(9,1)$ Majorana-Weyl spinors has been given for the first time in this paper and a proof that (1.1) is complete can be found in appendix B. As can easily be seen, there are two classes of stability subgroups characterized by their topology. Moreover the isotropy group of 9 or more spinors is $\{1\}$. Therefore backgrounds with more than 8 parallel spinors necessarily have $\hat{R}=0$.

The dilatino Killing spinor equation is not amenable to such a straightforward Lie algebraic interpretation. This has been one of the obstacles to find the geometry of all supersymmetric type I backgrounds. Nevertheless much progress has been made to systematically understand the geometry of supersymmetric type I backgrounds. In [28], the Killing spinor equations of type I supergravities have been solved, using the spinorial geometry method of [29], under the assumption that all the $\hat{\nabla}$-parallel spinors are Killing, i.e. all solutions of the gravitino Killing spinor equation are also solutions the dilatino one, see also [30] for an application to the common sector.

The supersymmetric backgrounds with $\hat{R}=0$ have been examined in [31]. In particular, $\hat{R}=0$ and $dH=0$ imply that the spacetime is a Lorentzian metric Lie group.

[1]Similar geometries appear in the context of (1 + 1)- and (1+0)-dimensional supersymmetric sigma models, see e.g. [22–25].

[2]The isotropy groups of spinors are representation sensitive. There are many more isotropy groups that appear for $\mathrm{Spin}(9,1)$ Majorana spinors, and the $\mathrm{Spin}(10,1)$ Majorana spinors have different isotropy groups.

These groups have been classified in [32, 31], based on some earlier work on Lorentzian Lie groups [33]. So, the class of supersymmetric backgrounds that remains to be examined is that for which some of the $\hat{\nabla}$-parallel spinors do not solve the dilatino Killing spinor equation and $\hat{R} \neq 0$.

In this paper, we classify the geometry of all supersymmetric type I backgrounds. This is done by completing the program, i.e. by solving the Killing spinor equations for those backgrounds for which only some of the $\hat{\nabla}$-parallel spinors solve the dilatino Killing spinor equation. We shall find that the Killing spinor equations allow for backgrounds for any $N \leq 8$. We have carried out the classification using a combination of the spinorial geometry method of [29] and its recent adaptation to nearly maximally supersymmetric backgrounds proposed in [34, 35]. The first part of the task is to find the $\hat{\nabla}$-parallel spinors and to solve the gravitino Killing spinor equation. This has been done in [28] and the parallel spinors have been identified in most cases. We give the parallel spinors of the $SU(2) \ltimes \mathbb{R}^8$ and $U(1) \ltimes \mathbb{R}^8$ cases that have not been included in [28].

Next it remains to identify the Killing spinors of a supersymmetric background, i.e. those $\hat{\nabla}$-parallel spinors that solve the dilatino Killing spinor equation as well. Clearly for a background with N Killing spinors and L $\hat{\nabla}$-parallel spinors, $1 \leq N \leq L$. The backgrounds with $N < L$ are referred as "*descendants*". The N Killing spinors of a supersymmetric background can span any N-plane in the L-dimensional vector space $\mathcal{P}$ of $\hat{\nabla}$-parallel spinors. Generically there are infinitely many choices of N-planes in an L-dimensional vector space, so at first sight it appears that the program cannot be carried out. However, according to the spinorial geometry method of [29], the Killing spinors should be identified up to a gauge transformation of the Killing spinor equations. So not all choices of N-planes give rise to different spacetime geometries. In particular any two N-planes that are related by a $Spin(9,1)$ transformation which *preserves the space of parallel spinors* give rise to the same spacetime geometry and fluxes up to a Lorentz transformation. Given that the solutions $(\epsilon_1, \dots, \epsilon_L)$ of the gravitino Killing spinor equation span $\mathcal{P}$, $\hat{\nabla}\epsilon_i = 0$, we shall identify the Killing spinors up to transformations of the group

$$\Sigma(\mathcal{P}) = \mathrm{Stab}(\mathcal{P})/\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)\,, \tag{1.2}$$

where $\mathrm{Stab}(\mathcal{P})$ is the subgroup of $Spin(9,1)$ which preserves the L-dimensional vector space $\mathcal{P}$ and $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$ is the subgroup of $Spin(9,1)$ which preserves each $(\epsilon_1, \dots, \epsilon_L)$ individually. We shall see that $\Sigma(\mathcal{P})$ is the product of a *Spin* group and an R-symmetry group of an appropriate lower-dimensional supergravity. It turns out that after an appropriate identification using $\Sigma(\mathcal{P})$, there is a finite number of distinct N-planes, for $N \leq L/2$, and so the classification can be completed. In particular the Killing spinors of all the descendants can be identified. For $N > L/2$, one can use a similar argument to specify the normals to the Killing spinors. As in [34], these can be used to determine the Killing spinors. The dilatino Killing spinor equation can again be solved.

The Killing spinors of all descendants can be characterized by two groups. One is the isotropy group of the parallel spinors we have already mentioned. The other is the isotropy group of the Killing spinors $\mathrm{Stab}_\Sigma(\epsilon_1, \dots, \epsilon_N)$ in $\Sigma(\mathcal{P})$. In the description of Stab_Σ, it is sufficient to consider $N \leq L/2$. This is because for $N > L/2$, it is more convenient to consider the analogous groups for the normals to the Killing spinors. However, these coincide with those of the Killing spinors for $N \leq L/2$.

A special case are the descendants of backgrounds with $L = 16$ parallel spinors. These backgrounds are parallelizable, $\hat{R} = 0$. In this case, our method reproduces the results of [31]. A summary of the geometric properties of all cases can be found at the conclusions.

We also investigate the conditions on the descendants imposed by the Bianchi identities and field equations of the type I supergravities. For $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$ non-compact, $L \leq 8$, it turns out that $dH = 0$ and the field equations of type I supergravities imply that the descendants exist if and only if the holonomy of $\hat{\nabla}$, $\mathrm{hol}(\hat{\nabla})$, reduces to a proper subgroup of $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$, i.e. $\mathrm{hol}(\hat{\nabla}) \subset \mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$. In particular, if one insists that $\mathrm{hol}(\hat{\nabla}) = \mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$, then under the same conditions, the gravitino Killing spinor equation implies the dilatino one and so the only backgrounds that occur are those for which $N = L$, i.e. those investigated in [28]. For $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$ compact, there are descendants for which $\mathrm{hol}(\hat{\nabla}) = \mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$. Moreover, the gravitino Killing spinor equation implies the dilatino one provided some conditions are satisfied in addition to those implied by $dH = 0$, the field equations and $\mathrm{hol}(\hat{\nabla}) = \mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$.

The gaugino Killing spinor equation, $F\epsilon = 0$, can also be understood in a similar way to that of the gravitino Killing spinor equation. In particular, the spacetime indices of the gauge field strength F can be interpreted as taking values in $\mathfrak{spin}(9,1)$, so either the spinors ϵ have a non-trivial stability subgroup in $\mathrm{Spin}(9,1)$ or the gauge connection is flat, $F = 0$. We shall not present a detailed analysis of the conditions on F implied by the dilatino Killing spinor equation. This is because the geometry of spacetime is not affected by the solutions of the gaugino Killing spinor equation.[3] Of course one can consider the possibility that some of the solutions of the gravitino and dilatino Killing spinor equations solve the gaugino one as well. However, it is more usual to take that either all parallel spinors solve the gaugino Killing spinor equation or that all solutions of the gravitino and dilatino Killing spinor equations also solve the gaugino one. In all cases the solutions of the gaugino Killing spinor equation can be deduced from those of the gravitino Killing spinor equation.

This paper is organized as follows: In section two, we describe how the gauge symmetry of the Killing spinors can be used to identify the Killing spinors of all supersymmetric type I backgrounds. In sections three to five, we investigate the descendants of $SU(4) \ltimes \mathbb{R}^8$-, $Sp(2) \ltimes \mathbb{R}^8$- and $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$-invariant parallel spinors and compare their geometry to that of backgrounds for which all parallel spinors are Killing in each case. In sections six and seven, we solve the Killing spinor equations, and those of their descendants, of backgrounds with $SU(2) \ltimes \mathbb{R}^8$- and $U(1) \ltimes \mathbb{R}^8$-invariant parallel spinors. In section eight, we examine the Killing spinor equations of the descendants of $\mathbb{R}^8$-parallel spinors. In section nine, we use the Bianchi identities and the field equations to investigate the conditions

[3]The gauge field may contribute in the modification of the Bianchi identity of H due to the anomaly cancellation mechanism, and so it affects the spacetime geometry only in the case that $dH \neq 0$. However to lowest order in α', $dH = 0$. If the anomaly correction is included, then the sigma model two-loop contribution to the field equations should be taken into account, see e.g. [10]. In any case, most of our analysis is independent of such assumptions on dH.

under which the holonomy of $\hat{\nabla}$ reduces to a subgroup of the isotropy group of the parallel spinors. We also present some applications. In sections ten, eleven and twelve, we solve the Killing spinor equations of the descendants of backgrounds with G_2-, SU(3)- and SU(2)-invariant parallel spinors, respectively. We also investigate the reduction of the holonomy and its consequences in each case. In section thirteen, we investigate the parallelizable backgrounds using the methods developed in this paper and confirm the results of [32, 31], and in section fifteen we give our conclusions. In appendix A, we summarize some aspects of the geometry of manifolds which admits $\hat{\nabla}$-parallel spinors and outline some of their geometric properties. In appendix B, we show that the list presented in (1.1) is complete, and in appendix C, we summarize some results on a group representation that we have used to investigate an $N = 4$ descendant of SU(2)-invariant parallel spinors. In appendix D, we give the additional parallel spinor bi-linears for the $\mathrm{SU}(2) \ltimes \mathbb{R}^8$ and $\mathrm{U}(1) \ltimes \mathbb{R}^8$ cases.

2. Preliminaries

The Killing spinor equations of type I and heterotic supergravities are

$$\mathcal{D}(e,H)_A\epsilon = \hat{\nabla}_A\epsilon = 0\,, \quad \mathcal{A}(e,H,\Phi)\epsilon = (\Gamma^A\partial_A\Phi - \frac{1}{12}H_{ABC}\Gamma^{ABC})\epsilon = 0 \tag{2.1}$$

where e is a frame, Φ is the dilaton, H is the NS⊗NS three-form field strength and

$$\hat{\nabla}_B Y^A = \nabla_B Y^A + \frac{1}{2}H^A{}_{BC}Y^C\,, \tag{2.2}$$

is a metric connection with torsion H. The spinors ϵ are in the positive chirality Majorana-Weyl representation S^+ of $\mathrm{Spin}(9,1)$ which in the conventions of [28] are represented by even-degree forms. (We use the conventions of [28] throughout this paper.)

The Lie subgroups of $\mathrm{Spin}(9,1)$ that leave some spinors invariant have been listed in (1.1). We collectively denote them with $\mathrm{Stab}(\epsilon_1,\ldots,\epsilon_L)$ for $L = 1,2,3,4,5,6,8$ and 16. These stability subgroups have the property that they leave every individual spinor invariant. Since the holonomy of $\hat{\nabla}$ is contained in $\mathrm{Spin}(9,1)$, the gravitino Killing spinor equation has solutions provided that

$$\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{Stab}(\epsilon_1,\ldots,\epsilon_L)\ . \tag{2.3}$$

If $\mathrm{Stab}(\epsilon) = \{1\}$, then the curvature of $\hat{\nabla}$ vanishes, $\hat{R} = 0$. This together with the closure of H, $dH = 0$, imply that the spacetime is a Lorentzian metric Lie group.

To solve the dilatino Killing spinor equation, we first assume that we have a solution of the gravitino Killing spinor equation, i.e. we have a given number of parallel spinors spanning a subspace $\mathcal{P}$ in the space of spinors S^+. Then we try to find the conditions for which some of the parallel spinors solve the dilatino Killing spinor equation as well. If N is the number of Killing spinors, then necessarily they are at most as many as the parallel $\hat{\nabla}$-spinors, so

$$N \leq \dim\mathcal{P} = L\ . \tag{2.4}$$

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Let $\{\eta_i\}$ be a basis in the space of parallel spinors, $\hat{\nabla}\eta_i = 0$, $\mathcal{P} = \mathbb{R} < \eta_1, \ldots, \eta_L >$. The Killing spinors can now be written as

$$\epsilon_r = \sum_{i=1}^{L} f_{ri}\eta_i \,, \tag{2.5}$$

where f is a matrix of spacetime functions of rank N. Since ϵ_r must remain $\hat{\nabla}$-parallel and $\{\eta_i\}$ is a basis, it is easy to show that in fact f is a constant matrix. Let $\mathcal{K}$ be the N-plane in $\mathcal{P}$ spanned by the Killing spinors, $\mathcal{K} = \mathbb{R} < \epsilon_1, \ldots \epsilon_N >$.

Next suppose that $\ell \in \mathrm{Spin}(9,1)$ and that it preserves[4] $\mathcal{P}$, $\ell\mathcal{P} \subseteq \mathcal{P}$. Then consider $\ell\epsilon_r$ and observe that

$$\begin{aligned} \mathcal{D}(e^{\ell^{-1}}, H^{\ell^{-1}})\ell\epsilon_r = \ell\mathcal{D}(e,H)\epsilon_r = 0\,, \\ \mathcal{A}(e^{\ell^{-1}}, H^{\ell^{-1}}, \Phi^{\ell^{-1}})\ell\epsilon_r = \ell\mathcal{A}(e,H,\Phi)\epsilon_r = 0\,, \end{aligned} \tag{2.6}$$

where $e^{\ell^{-1}}, H^{\ell^{-1}}, \Phi^{\ell^{-1}}$ are the Lorentz transformed frame, H and dilaton with respect to the inverse ℓ^{-1} Lorentz transformation[5] associated with $\ell \in \mathrm{Spin}(9,1)$. Therefore the spinors $\ell\epsilon_r$ are also solutions of the Killing spinor equations up to a Lorentz rotation of the frame and the fluxes. Since we identify backgrounds related by frame Lorentz transformations, one concludes that the N-planes $\mathcal{K}$ and $\ell\mathcal{K}$ give rise to the same spacetime geometry and fluxes. Thus to classify the N-supersymmetric backgrounds, it is sufficient to find all N-planes in $\mathcal{P}$ up to transformations in $\mathrm{Spin}(9,1)$ that preserve $\mathcal{P}$.

To continue we have to identify the subgroup $\Sigma(\mathcal{P}) \subseteq \mathrm{Spin}(9,1)$ which preserves $\mathcal{P}$, where $\mathcal{P} = \mathbb{R} < \eta_1, \ldots, \eta_L >$. As we have mentioned in the introduction, first define the stability subgroup of $\mathcal{P}$ as

$$\mathrm{Stab}(\mathcal{P}) = \{\ell \in \mathrm{Spin}(9,1) \text{ s.t. } \ell\mathcal{P} \subset \mathcal{P}\} \tag{2.7}$$

Clearly $\mathrm{Stab}(\eta_1, \ldots \eta_L) \subseteq \mathrm{Stab}(\mathcal{P})$. In fact $\mathrm{Stab}(\eta_1, \ldots \eta_L)$ is a normal subgroup. Then we define

$$\Sigma(\mathcal{P}) = \mathrm{Stab}(\mathcal{P})/\mathrm{Stab}(\eta_1, \ldots \eta_L)\ . \tag{2.8}$$

$\Sigma(\mathcal{P})$ may act non-trivially on the space of parallel spinors preserving the subspace spanned by them and takes the rôle of the gauge group in the context of the spinorial geometry approach to solving the Killing spinor equations. The groups $\Sigma(\mathcal{P})$ are summarized in table 1. It can be easily seen that they are products of the type $\Sigma(\mathcal{P}) = \mathrm{Spin}(d,1) \times R$. Such groups are reminiscent of the gauge groups of $(d+1)$-supergravities, where R is the R-symmetry group. It may be possible to make this correspondence more precise by considering compactifications of type I supergravity on supersymmetric backgrounds with an appropriate $\Sigma(\mathcal{P})$ group.

[4]The assumption $\ell\mathcal{P} \subseteq \mathcal{P}$ can be relaxed but it is more convenient to consider only those ℓ that preserve $\mathcal{P}$.

[5]We denote with the same symbol the element $\ell \in \mathrm{Spin}(9,1)$ and its projection on the Lorentz group.

L	$Stab(\epsilon_1, \dots, \epsilon_L)$	$\Sigma(\mathcal{P})$
1	$\mathrm{Spin}(7) \ltimes \mathbb{R}^8$	$\mathrm{Spin}(1,1)$
2	$\mathrm{SU}(4) \ltimes \mathbb{R}^8$	$\mathrm{Spin}(1,1) \times \mathrm{U}(1)$
3	$\mathrm{Sp}(2) \ltimes \mathbb{R}^8$	$\mathrm{Spin}(1,1) \times \mathrm{SU}(2)$
4	$(\mathrm{SU}(2) \times \mathrm{SU}(2)) \ltimes \mathbb{R}^8$	$\mathrm{Spin}(1,1) \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$
5	$\mathrm{SU}(2) \ltimes \mathbb{R}^8$	$\mathrm{Spin}(1,1) \times \mathrm{Sp}(2)$
6	$\mathrm{U}(1) \ltimes \mathbb{R}^8$	$\mathrm{Spin}(1,1) \times \mathrm{SU}(4)$
8	$\mathbb{R}^8$	$\mathrm{Spin}(1,1) \times \mathrm{Spin}(8)$
2	G_2	$\mathrm{Spin}(2,1)$
4	$\mathrm{SU}(3)$	$\mathrm{Spin}(3,1) \times \mathrm{U}(1)$
8	$\mathrm{SU}(2)$	$\mathrm{Spin}(5,1) \times \mathrm{SU}(2)$
16	$\{1\}$	$\mathrm{Spin}(9,1)$

Table 1: In the columns are the numbers of parallel spinors, their isotropy groups and the $\Sigma(\mathcal{P})$ groups, respectively. The $\Sigma(\mathcal{P})$ groups are a product of a *Spin* group and an R-symmetry group.

To see how $\Sigma(\mathcal{P})$ is used, let us first choose $\mathcal{P}$ and suppose that only one of the $\hat{\nabla}$-parallel spinors also solves the dilatino Killing spinor equation, say ϵ and so $N = 1$. The spinor ϵ can be expressed as a linear combination of a basis of parallel spinors $\epsilon = f_i \eta_i$. As we have explained, ϵ and $\ell\epsilon$, $\ell \in \Sigma(\mathcal{P})$, give rise to the same spacetime geometry. So the Killing spinors which may lead to different spacetime geometries are labeled by the orbits, $\mathcal{O}_{\Sigma(\mathcal{P})}(\mathcal{P})$, of $\Sigma(\mathcal{P})$ in $\mathcal{P}$. Hence, to find all $N = 1$ backgrounds with L $\hat{\nabla}$-parallel spinors, it suffices to choose a single representative from each orbit of $\Sigma(\mathcal{P})$ in $\mathcal{P}$. In general these representatives depend on as many different parameters as the number of deformations that preserve the orbit. In particular the representatives of generic orbits, i.e. orbits of maximal co-dimension, the number of parameters is equal to the co-dimension of the orbit in $\mathcal{P}$. The Killing spinor equations are linear, so the Killing spinor is specified up to an overall scale. As a result, the number of independent parameters that the Killing spinor depends on is at most the dimension of the deformations that preserve the associated orbit. The number of parameters of representatives of generic orbits is either $\mathrm{codim}\,\mathcal{O}_{\Sigma(\mathcal{P})}(\mathcal{P})$ or $\mathrm{codim}\,\mathcal{O}_{\Sigma(\mathcal{P})}(\mathcal{P}) - 1$ depending on whether $\Sigma(\mathcal{P})$ contains a scale generator. In most generic cases, it turns out that $\mathrm{codim}\,\mathcal{O}$ is either zero or one and so we have to specify a single direction.

To continue, we proceed inductively. Let $\mathcal{K}$ be the N-plane in $\mathcal{P}$ spanned by the first N Killing spinors,

$$0 \to \mathcal{K} \to \mathcal{P} \to \mathcal{P}/\mathcal{K} \to 0\,, \tag{2.9}$$

$\mathcal{K} = \mathbb{R} < \epsilon_1, \dots, \epsilon_N >$. To choose the $(N+1)$-th Killing spinor, we first consider $\mathrm{Stab}(\mathcal{K}) \subseteq \Sigma(\mathcal{P})$ that preserves $\mathcal{K}$, i.e.

$$\mathrm{Stab}(\mathcal{K}) = \{\ell \in \Sigma(\mathcal{P})\,, \quad \ell\mathcal{K} \subseteq \mathcal{K}\}\ . \tag{2.10}$$

The strategy we adopt is to use $\mathrm{Stab}(\mathcal{K})$ to choose the $(N+1)$-th Killing spinor. For this, first choose a spinor ϵ_{N+1} which is linearly independent from those in $\mathcal{K}$. Since the

Killing spinor equations are linear, it suffices to choose ϵ_{N+1} up to elements in $\mathcal{K}$. Thus ϵ_{N+1} can be thought of as an element in $\mathcal{P}/\mathcal{K}$. Moreover $\mathrm{Stab}(\mathcal{K})$ acts on $\mathcal{P}/\mathcal{K}$ preserving the plane of the first N Killing spinors. Using again the identification of supersymmetric backgrounds under frame Lorentz rotations, the $(N+1)$-th Killing spinor ϵ_{N+1} can be chosen to be a representative of the orbits $\mathcal{O}_{\mathrm{Stab}(\mathcal{K})}(\mathcal{P}/\mathcal{K})$ of $\mathrm{Stab}(\mathcal{K})$ in $\mathcal{P}/\mathcal{K}$. Again the number of independent parameters that ϵ_{N+1} has depends on the type of orbit it represents. If it is a generic orbit, the number of parameters is either $\mathrm{codim}\,\mathcal{O}_{\mathrm{Stab}(\mathcal{K})}(\mathcal{P}/\mathcal{K})$ or $\mathrm{codim}\,\mathcal{O}_{\mathrm{Stab}(\mathcal{K})}(\mathcal{P}/\mathcal{K})-1$ depending on whether $\mathrm{Stab}(\mathcal{K})$ acts with or without a scale transformation on $\mathcal{P}/\mathcal{K}$.

The above described procedure works well for all $1\leq N\leq L/2$. For $N>L/2$ in some cases it is more convenient, instead of determining the Killing spinors up to $\mathrm{Spin}(9,1)$ transformations, to specify their normals. For this first recall that we have chosen $\mathcal{P}\subseteq S^+$, where S^+ is the positive chirality Majorana-Weyl representation of $\mathrm{Spin}(9,1)$. The dual of $(S^+)^\star$ is identified with S^-, the negative chirality Majorana-Weyl spinors, via the Majorana *Pin*-invariant inner product B, i.e. $S^-=B((S^+)^\star)$, see [28, 34] for details. Define $\mathcal{Q}=B(\mathcal{P}^\star)$. Next, $\mathrm{Spin}(9,1)$ acts on S^- and so as before define $\Sigma(\mathcal{Q})$. The hyperplane of Killing spinors of $N=L-1$ supersymmetric backgrounds has a unique normal in $\mathcal{Q}$. Using the identification of supersymmetric backgrounds under frame Lorentz transformations and an argument as above, backgrounds with distinct geometries are labeled by the orbits $\mathcal{O}_{\Sigma(\mathcal{Q})}(\mathcal{Q})$, i.e. by the choice of the normal ν up to gauge transformations that preserve $\mathcal{Q}$. The normal spinor is specified up to an overall scale, i.e. we need to specify only the normal direction. Thus the number of parameters that the normal spinor has depends on the type of orbit it represents. For generic orbits, the number of parameters is either $\mathrm{codim}\mathcal{O}_{\Sigma(\mathcal{Q})}(\mathcal{Q})$ or $\mathrm{codim}\mathcal{O}_{\Sigma(\mathcal{Q})}(\mathcal{Q})-1$ depending on the way that $\Sigma(\mathcal{Q})$ acts on $\mathcal{Q}$. Then the hyperplane of the Killing spinors is specified by the orthogonality condition

$$B(\nu,\mathcal{K})=0\;. \tag{2.11}$$

To continue, one proceeds inductively. First define $\mathcal{N}$ as the $(L-N)$-plane in $\mathcal{Q}$ spanned by the first $L-N$ normal spinors. To specify an additional Killing spinor up to a $\mathrm{Spin}(9,1)$ transformation, define the subgroup $\mathrm{Stab}(\mathcal{N})\subseteq\mathrm{Spin}(9,1)$ that preserves $\mathcal{N}$ and consider the sequence

$$0\rightarrow\mathcal{N}\rightarrow\mathcal{Q}\rightarrow\mathcal{Q}/\mathcal{N}\rightarrow 0\;. \tag{2.12}$$

The $(L-N+1)$ normal spinor ν_{L-N+1} is chosen to be linearly independent from the first $L-N$ normal spinors and it is specified up to elements in $\mathcal{N}$. This is because the $(N-1)$-Killing spinors will span a hyperplane in $\mathcal{K}$ and so they will be always orthogonal to $\mathcal{N}$. Thus ν_{L-N+1} can be thought of as an element in $\mathcal{Q}/\mathcal{N}$. Using the identification of supersymmetric backgrounds under frame Lorentz transformations again, the additional normal spinor can be chosen as a representative of the orbits $\mathcal{O}_{\mathrm{Stab}(\mathcal{N})}(\mathcal{Q}/\mathcal{N})$. The number of independent parameters of the new normal depend on the type of orbit it represents. For generic orbits, the number of parameters is either $\mathrm{codim}\,\mathcal{O}_{\mathrm{Stab}(\mathcal{N})}(\mathcal{Q}/\mathcal{N})$ or $\mathrm{codim}\,\mathcal{O}_{\mathrm{Stab}(\mathcal{N})}(\mathcal{Q}/\mathcal{N})-1$. In turn the Killing spinors are determined by the orthogonality

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condition

$$B(\mathcal{N}, \mathcal{K}) = 0\,, \tag{2.13}$$

where now $\mathcal{N}$ is spanned by all $N - L + 1$ normal spinors. As we have mentioned in the introduction, we refer to the backgrounds with N supersymmetries, $N < L$, that arise from a given set of $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$-invariant parallel spinors as "descendants" of $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L)$.

3. The descendants of SU(4) $\ltimes \mathbb{R}^8$

A (complex) basis[6] in the space of parallel spinors can be chosen as

$$1\ . \tag{3.1}$$

Observe that $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times U(1)$, where the generator of $\mathrm{Spin}(1,1)$ is Γ^{+-} and the generator of $U(1)$ can be chosen as $i\Gamma^{1\bar{1}}$. Observe that $\mathrm{Spin}(1,1) = \mathbb{R}^*$. There is a single descendant background with $N = 1$ supersymmetry. The dilatino Killing spinor equation can be written as

$$\mathcal{A}(1 + e_{1234}) = 0\ . \tag{3.2}$$

We use the conventions of [28] to denote the spinors and forms that arise in the analysis that follows. Note that the stability subgroup of the Killing spinor in $\Sigma(\mathcal{P})$ is $\mathrm{Stab}_\Sigma(1+e_{1234}) = \{1\}$.

The strategy we adopt to organize the solutions of the Killing spinor equations in all cases is to first solve the gravitino Killing spinor equation. The conditions that arise are the same for all descendants. Then the dilatino Killing spinor equation is solved for each descendant and the solutions are expressed in representations of $\mathrm{Stab}(\epsilon_1, \dots \epsilon_L)$, i.e. the isotropy group of the parallel spinors.

3.1 Geometry of the gravitino Killing spinor equation

The solution of the gravitino Killing spinor equation can be read off from the results of [28]. The conditions that this imposes on the geometry is that $\mathrm{hol}(\hat{\nabla}) \subseteq SU(4) \ltimes \mathbb{R}^8$. This is equivalent to requiring that the spacetime admits the $\hat{\nabla}$-parallel forms

$$e^-, \quad e^- \wedge \omega_I\,, \quad e^- \wedge \mathrm{Re}\,\chi\,, \quad e^- \wedge \mathrm{Im}\,\chi\,, \tag{3.3}$$

where

$$\omega_I = -(e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9)\,, \quad \chi = (e^1 + ie^6) \wedge \cdots \wedge (e^4 + ie^9)\,, \tag{3.4}$$

and I is an endomorphism constructed by the metric and the two form ω_I. In particular I can be thought of as an "almost complex" structure in the "transverse space" to the

[6] The associated real basis is $\big(1 + e_{1234}, i(1 - e_{1234})\big)$. This can be easily found by taking the real and imaginary parts of the complex spinor 1 with respect to a reality condition that defines the Majorana-Weyl representation of $\mathrm{Spin}(9,1)$, see [28].

light-cone directions. In the Hermitian light-cone frame $e^+, e^-, e^\alpha, e^{\bar\alpha}$, it has components $I^\alpha{}_\beta = i\delta^\alpha{}_\beta$, $\alpha, \beta = 1, 2, 3, 4$.

To continue, the metric and three-form can be written as in appendix A. In this case, $\mathfrak{k} = \mathfrak{su}(4)$. So, $\mathfrak{su}(4)^\perp$ is spanned by the (2,0)- and (0,2)-forms, and ω_I in $\Lambda^2(\mathbb{R}^8) \otimes \mathbb{C}$. As we have explained in appendix A, the components of $H^{\mathfrak{su}(4)^\perp}_{-ij}$ are determined by the geometry. In particular, one finds that

$$\begin{aligned} H^{2,0+0,2}_{-ij} &= -\frac{1}{2}[i_I(\nabla_-\omega)]_{ij} = \frac{1}{2\cdot 3!}[(\nabla_-\mathrm{Re}\,\chi)_{ik_1k_2k_3}\mathrm{Re}\,\chi_j{}^{k_1k_2k_3}]^{2,0+0,2} \\ &= \frac{1}{2\cdot 3!}[(\nabla_-\mathrm{Im}\,\chi)_{ik_1k_2k_3}\mathrm{Im}\,\chi_j{}^{k_1k_2k_3}]^{2,0+0,2}\,, \\ H_{-ij}\omega_I^{ij} &= \frac{1}{2\cdot 4!}(\nabla_-\mathrm{Re}\,\chi)_{k_1k_2k_3k_4}\mathrm{Im}\chi^{k_1k_2k_3k_4}\,, \quad i,j,\dots = 1,2,3,4,6,7,8,9. \end{aligned} \tag{3.5}$$

Furthermore, the conditions along the transverse directions give

$$\begin{aligned} H^{\mathrm{rest}} &= \frac{1}{3!}H_{ijk}e^i\wedge e^j\wedge e^k \\ &= -i_I\tilde d\omega_I - 2\mathcal{N}(I) = \star(\tilde d\omega_I\wedge\omega_I) - \frac{1}{2}\star(\theta_{\omega_I}\wedge\omega_I\wedge\omega_I) + \mathcal{N}(I), \end{aligned} \tag{3.6}$$

where $\tilde d$ denotes the exterior derivative projected along the eight directions transverse to the light-cone and the Hodge duality $\star$ operation[7] is taken with volume form $d\mathrm{vol} = e^1\wedge\cdots\wedge e^4\wedge e^6\wedge\cdots\wedge e^9$. For a similar expression for H^{rest} in the context of Riemannian geometry see [9]. In addition $\theta_{\omega_I} = -\star(\star\tilde d\omega_I\wedge\omega_I)$ is the Lee form of ω_I, and $\mathcal{N}(I)$ is a (3,0) and (0,3) tensor, the Nijenhuis tensor of the endomorphism I, $\mathcal{N}(I)_{\alpha\beta\gamma} = 4H_{\alpha\beta\gamma}$. It remains to find the conditions on the geometry. It turns out that

$$\begin{aligned} (de_+)^{2,0+0,2}_{ij} &= -\frac{1}{2}[i_I(\nabla_+\omega)]_{ij}\,, \\ (de_+)_{ij}\omega^{ij} &= \frac{1}{2\cdot 4!}(\nabla_+\mathrm{Re}\,\chi)_{k_1k_2k_3k_4}\mathrm{Im}\chi^{k_1k_2k_3k_4}\,, \\ W_2 &= 0\,, \quad \theta_\omega = \theta_{\mathrm{Re}\,\chi}\,, \end{aligned} \tag{3.7}$$

where $de^- = \eta^{-+}de_+$ and $\theta_{\mathrm{Re}\,\chi} = -\frac{1}{4}\star(\star\tilde d\mathrm{Re}\,\chi\wedge\mathrm{Re}\,\chi)$ is the Lee form of $\mathrm{Re}\,\chi$. The first two conditions are required for the compatibility of determining $H^{\mathfrak{k}^\perp}_{+ij}$ in terms of both the Lie derivative of ω and χ along the parallel vector field. The last two conditions, are required for the existence of H^{rest}. W_2 is one of the Gray-Hervella classes for determining $\mathrm{U}(n)$ structures [42]. The vanishing of W_2 implies that that the Nijenhuis tensor is skew-symmetric in all three indices. The non-vanishing of the Nijenhuis tensor indicates that the endomorphism I is not integrable. The equality of the Lee forms θ_ω and $\theta_{\mathrm{Re}\chi}$ can also be expressed as a condition on SU(4) classes by saying that $W_4 = W_5$.

3.2 Geometry of $N = 1$ supersymmetric backgrounds

The solution of the dilatino Killing spinor equation is

$$\begin{aligned} \partial_+\Phi = 0\,, \quad H_{+\alpha}{}^\alpha = 0\,, \quad &- H_{+\bar\alpha_1\bar\alpha_2} + \frac{1}{2}H_{+\beta_1\beta_2}\epsilon^{\beta_1\beta_2}{}_{\bar\alpha_1\bar\alpha_2} = 0\,, \\ \partial_{\bar\alpha}\Phi + \frac{1}{6}H_{\beta_1\beta_2\beta_3}\epsilon^{\beta_1\beta_2\beta_3}{}_{\bar\alpha} &- \frac{1}{2}H_{\bar\alpha\beta}{}^\beta - \frac{1}{2}H_{-+\bar\alpha} = 0\,. \end{aligned} \tag{3.8}$$

[7]Note that $\star\psi_{i_1\dots i_{n-k}} = \frac{1}{k!}\psi_{j_1\dots j_k}\epsilon^{j_1\dots j_k}{}_{i_1\dots i_{n-k}}$.

This is the same as that which has been found in [28] for $N = 1$ supersymmetric Spin(7) $\ltimes \mathbb{R}^8$ backgrounds. The above conditions are in addition to those we have stated in the previous section for the existence of a solution to the gravitino Killing spinor equation. In particular, (3.8) can be rewritten as

$$\begin{aligned}
&\partial_+\Phi = 0\,, \quad de^- \in \mathfrak{spin}(7) \oplus_s \mathbb{R}^8\,, \\
&(d\Phi)_i + \frac{1}{8\cdot 3!}\mathcal{N}_{k_1k_2k_3}(\mathrm{Re}\,\chi)^{k_1k_2k_3}{}_i - \frac{1}{2}(\theta_{\omega_I})_i - \frac{1}{2}H_{-+i} = 0\,,
\end{aligned} \tag{3.9}$$

The space of spacetime two-forms decomposes under the action of Spin(7) $\ltimes \mathbb{R}^8$ into irreducible representations. The condition $de^- \in \mathfrak{spin}(7) \oplus_s \mathbb{R}^8$ means that the two-form de^- takes values in the $\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$ subspace. This is equivalent to writing $de^- = \alpha + e^- \wedge \beta$, where α is a two-form with values in the **21** irreducible representation in the decomposition of the transverse two-forms in Spin(7) representations and β is a transverse one-form. Alternatively, this condition can be written as

$$(de^-)_{ij}\omega^{ij} = 0\,, \quad (de^-)_{ij} = \frac{1}{4}(de^-)_{kl}\,\mathrm{Re}\,\chi_{ij}{}^{kl}\,. \tag{3.10}$$

Both these conditions can be thought of as additional conditions on the geometry of spacetime. The components of de^- that lie in $\mathfrak{su}^\perp(4) \subset \mathfrak{spin}(7)$ are not required to vanish. The components of de^- along $\mathfrak{su}(4) \oplus_s \mathbb{R}^8$ are not restricted by the Killing spinor equations. The last condition in (3.9) is a generalization of the conformal balance condition that it is well-known for some supersymmetric type I backgrounds, see e.g. [7]. Variations of this condition appear in all solutions of dilatino Killing spinor equation for all descendants.

3.3 Comparison with N=2

It is instructive to compare the conditions we have found for the $N = 1$ backgrounds with the results of [28] for the $N = L = 2$ backgrounds. The solution of the dilatino Killing spinor equation is

$$\begin{aligned}
\partial_+\Phi = 0\,, \quad H_{+\alpha}{}^\alpha = 0\,, \quad H_{+\bar\alpha_1\bar\alpha_2} = H_{\beta_1\beta_2\beta_3} = 0\,, \\
\partial_{\bar\alpha}\Phi - \frac{1}{2}H_{\bar\alpha\beta}{}^\beta - \frac{1}{2}H_{-+\bar\alpha} = 0\,.
\end{aligned} \tag{3.11}$$

This can also be rewritten as

$$\begin{aligned}
\partial_+\Phi = 0\,, \quad \mathcal{N}(I)_{ijk} = 0\,, \quad de^- \in \mathfrak{su}(4) \oplus_s \mathbb{R}^8 \subset \mathfrak{spin}(7) \oplus_s \mathbb{R}^8\,, \\
(d\Phi)_i - \frac{1}{2}(\theta_{\omega_I})_i - \frac{1}{2}H_{-+i} = 0\,,
\end{aligned} \tag{3.12}$$

i.e. the endomorphism I is integrable and both $e^- \wedge \omega_I$ and $e^- \wedge \chi$ are invariant under the action of the $\hat\nabla$-parallel vector field e_+, i.e.

$$\mathcal{L}_{e_+}(e^- \wedge \omega_I) = \mathcal{L}_{e_+}(e^- \wedge \chi) = 0\,. \tag{3.13}$$

The conditions that arise from the gravitino Killing spinor equation are the same. The differences of $N = 1$ and $N = 2$ backgrounds are summarized in table 2. The isotropy group Stab_Σ of the $N = 1$ Killing spinor in $\Sigma(\mathcal{P})$ is also tabulated.

$\mathrm{SU}(4) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	Stab_Σ
$N=1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) \neq 0$	$\{1\}$
$N=2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = 0$	

Table 2: The differences in the geometry of $N=1$ and $N=2$ backgrounds are in the non-vanishing components of de^- and $\mathcal{N}(I)$. It is understood that the remaining conditions of the dilatino Killing spinor equation for $N=1$ backgrounds are valid.

4. The descendants of $\mathrm{Sp}(2) \ltimes \mathbb{R}^8$

A basis in the space of parallel spinors can be chosen as

$$1 + e_{1234}\,, \quad i(1 - e_{1234})\,, \quad i(e_{12} + e_{34})\,, \tag{4.1}$$

i.e. $\mathcal{P} = \mathbb{R} < 1 + e_{1234}, i(1 - e_{1234}), i(e_{12} + e_{34}) >$. It is easy to see that in this case $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times \mathrm{SU}(2)$, where $\mathrm{SU}(2)$ acts on $\mathcal{P}$ with the three-dimensional representation. In particular, in the basis given above $\mathfrak{su}(2)$ is spanned by $\Gamma^{\bar{1}\bar{2}} - \Gamma^{34}, \Gamma^{12} - \Gamma^{\bar{3}\bar{4}}, \frac{i}{2}(\Gamma^{1\bar{1}} + \Gamma^{2\bar{2}} + \Gamma^{3\bar{3}} + \Gamma^{4\bar{4}})$, and the generator of $\mathrm{Spin}(1,1)$ is Γ^{+-}. From these, it is straightforward to find the $N=1$ and $N=2$ descendants.

As in the previous case, we first solve the gravitino Killing spinor equation. The conditions that arise from the analysis of both the gravitino and dilatino Killing spinor equations in all cases can be most efficiently organized as conditions on two endomorphisms I and J. It turns out that this is a generic feature of all cases that have parallel spinors with non-compact isotropy groups. In every new case, we shall introduce an appropriate new endomorphism.

4.1 Geometry of the gravitino Killing spinor equation

The geometry of the gravitino Killing spinor equations can be investigated as in the $\mathrm{SU}(4) \ltimes \mathbb{R}^8$ case. The difference is that there are three $\hat{\nabla}$-parallel three-forms, instead of one, associated with the Hermitian forms of an almost hyper-complex structure. Let $\{I_r, r = 1,2,3\} = \{I, J, K\}$ be endomorphisms such that $I_r I_s = -\delta_{rs} \mathbf{1}_{8\times 8} + \epsilon_{rst} I_t$. Then if the forms

$$e^-\,, \quad e^- \wedge \omega_I\,, \quad e^- \wedge \omega_J\,, \tag{4.2}$$

are $\hat{\nabla}$-parallel, then $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{Sp}(2) \ltimes \mathbb{R}^8$, where ω_I, ω_J and ω_K are the associated Hermitian forms. One can easily show that $e^- \wedge \omega_K$ is parallel as well. In particular ω_I can be chosen as in the $\mathrm{SU}(4) \ltimes \mathbb{R}^8$ case while $\omega_J = 2\mathrm{Re}(e^1 \wedge e^2 + e^3 \wedge e^4)$.

The conditions on the geometry can be described as two copies of those of the $\mathrm{SU}(4) \ltimes \mathbb{R}^8$ case, with each copy associated with one of the endomorphisms I, J, that have to be valid simultaneously. To proceed, we have to identify the directions that lie in $\mathfrak{sp}(2)^\perp$, $\Lambda^2(\mathbb{R}^8) = \mathfrak{sp}(2) \oplus \mathfrak{sp}(2)^\perp$. For this first observe that $\mathfrak{sp}(2)$ is spanned by the (1,1)-forms in $\Lambda^2(\mathbb{R}^8) \otimes \mathbb{C}$ with respect to both I and J. Thus $\mathfrak{sp}(2)^\perp$ is spanned by the (2,0)- and

(0,2)-forms with respect to I, and those (1,1)-forms with respect to I that are (2,0) and (0,2) with respect to J. So if one sets $H_-^{\mathfrak{sp}(2)^\perp} = (H_-^{2,0+0,2}, \check{H}_-^{1,1})$, then one can write

$$\begin{aligned} H^{2,0+0,2}_{-ij} &= -\frac{1}{2}[i_I(\nabla_-\omega_I)]_{ij} \,, \\ \check{H}^{1,1}_{-ij} &= -\frac{1}{2}[i_J(\nabla_-\omega_J)]^{1,1}_{ij} \,, \end{aligned} \tag{4.3}$$

where the projections (2,0), (0,2) and (1,1) have been taken with respect to the I endomorphism. In addition, we get the geometric conditions

$$\begin{aligned} (de_+)^{2,0+0,2}_{ij} &= -\frac{1}{2}[i_I(\nabla_+\omega_I)]_{ij} \,, \\ (d\check{e}_+)^{1,1}_{ij} &= -\frac{1}{2}[i_J(\nabla_+\omega_J)]^{1,1}_{ij} \,. \end{aligned} \tag{4.4}$$

Furthermore, one finds that

$$\begin{aligned} H^{\rm rest} &= -i_I\tilde{d}\omega_I - 2\mathcal{N}(I) = \star(\tilde{d}\omega_I\wedge\omega_I) - \frac{1}{2}\star(\theta_{\omega_I}\wedge\omega_I\wedge\omega_I) + \mathcal{N}(I) \\ &= -i_J\tilde{d}\omega_J - 2\mathcal{N}(J) = \star(\tilde{d}\omega_J\wedge\omega_J) - \frac{1}{2}\star(\theta_{\omega_J}\wedge\omega_J\wedge\omega_J) + \mathcal{N}(J) \,. \end{aligned} \tag{4.5}$$

The equality involving the I and J expression should be interpreted as a condition on the geometry. Moreover $W_2(I) = W_2(J) = 0$ which is equivalent to the condition that the Nijenhuis tensor of both I and J is skew-symmetric.

4.2 N=1

The dilatino Killing spinor equation is

$$\mathcal{A}(1+e_{1234}) = 0 \,, \tag{4.6}$$

The solution has been given in (3.8) or equivalently (3.9).

4.3 N=2

The dilatino Killing spinor equation is

$$\mathcal{A}1 = 0 \,. \tag{4.7}$$

The solution to the dilatino Killing spinor equation has already been given in either (3.11) or equivalently (3.12).

4.4 Comparison with N=3

The conditions that arise form the dilatino Killing spinor equation in this case have been computed in [28] and can be summarized as

$$\begin{aligned} &\partial_+\Phi = 0 \,, \quad de^- \in \mathfrak{sp}(2)\oplus_s\mathbb{R}^8 \,, \quad \mathcal{N}(I)_{ijk} = \mathcal{N}(J)_{ijk} = 0 \,, \\ &\qquad\qquad 2\partial_i\Phi - H_{-+i} = (\theta_{\omega_I})_i = (\theta_{\omega_J})_i \,. \end{aligned} \tag{4.8}$$

The conditions on the geometry that arise from the gravitino Killing spinor equation are the same in all cases. The differences arise in the solution to the dilatino Killing spinor equation and have been summarized in table 3. We also give Stab_Σ.

$\mathrm{Sp}(2) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	θ	Stab_Σ
$N=1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I), \mathcal{N}(J) \neq 0$	$-$	$\mathrm{U}(1)$
$N=2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=0, \mathcal{N}(J) \neq 0$	$-$	
$N=3$	$\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=0$	$\theta_{\omega_I}=\theta_{\omega_J}$	

Table 3: The differences in the geometry of $N=1$, $N=2$ and $N=3$ backgrounds are in the non-vanishing components of de^-, and $\mathcal{N}(I)$ and $\mathcal{N}(J)$, and the relation between the Lee forms. $-$ indicates that there is no relation between the Lee forms. It is understood that the remaining conditions of the dilatino Killing spinor equation for $N=1$ backgrounds are valid.

N	Stab_Σ
1	$\mathrm{SU}(2)$
2	$\mathrm{U}(1)$

Table 4: The first column denotes the number of supersymmetries and the second column the stability subgroups of Killing spinors for $N \leq 2$ in $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times \mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R$.

5. The descendants of $(\mathrm{SU}(2) \times \mathrm{SU}(2)) \ltimes \mathbb{R}^8$

A complex basis in the space of $(\mathrm{SU}(2) \times \mathrm{SU}(2)) \ltimes \mathbb{R}^8$-invariant spinors is

$$1\,, \quad e_{12}\ . \tag{5.1}$$

It is easy to see that in this case $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times \mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R$. Identifying $\mathcal{P} = \mathbb{H}$, $\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R$ acts as

$$x \to a x \bar{b}\,, \quad x \in \mathbb{H}\,, \quad a \in \mathrm{Sp}(1)_L\,, \quad b \in \mathrm{Sp}(1)_R\ . \tag{5.2}$$

In addition, $\mathrm{Spin}(1,1)$ has the generator Γ^{+-}. There is a single type of orbit in $\mathcal{P}$ with stability subgroup $\mathrm{Sp}(1)$ acting with the three-dimensional representation on the remaining space. From this, one can easily determine the Killing spinors for all cases. The Stab_Σ groups are given in table 4.

5.1 Geometry of the gravitino Killing spinor equation

The gravitino Killing spinor equation implies that $\mathrm{hol}(\hat{\nabla}) \subseteq (\mathrm{SU}(2) \times \mathrm{SU}(2)) \ltimes \mathbb{R}^8$. In turn this is equivalent to requiring [28] that the forms

$$e^-\,, \quad e^- \wedge \omega_1\,, \quad e^- \wedge \omega_2\,, \quad e^- \wedge \chi_1\,, \quad e^- \wedge \chi_2\,, \tag{5.3}$$

are $\hat{\nabla}$-parallel, where $\omega_1 = -i(e^1 \wedge e^{\bar{1}} + e^2 \wedge e^{\bar{2}})$, $\omega_2 = -i(e^3 \wedge e^{\bar{3}} + e^4 \wedge e^{\bar{4}})$, $\chi_1 = 2e^1 \wedge e^2$, and $\chi_2 = 2e^3 \wedge e^4$. In this case $\mathfrak{k}^\perp$ is spanned by the (2,0) and (0,2) forms of the endomorphisms I, J and L, where $\omega_I = \omega_1 + \omega_2$, $\omega_J = \mathrm{Re}(\chi_1 + \chi_2)$ and $\omega_L = \omega_1 - \omega_2$. The endomorphisms satisfy the algebra

$$I^2 = J^2 = L^2 = -1_{8\times 8}\,, \quad IJ = -JI\,, \quad IL = LI\,, \quad JL = -LJ\ . \tag{5.4}$$

In addition $(I, J, K = IJ)$ and $(L, M = ILJ, N = -IJ)$ are almost hyper-complex structures, and $P = IL$ is an almost product structure. The geometric conditions that arise from the gravitino Killing spinor equation are those that arise from three $\mathrm{U}(4) \ltimes \mathbb{R}^8$ structures each associated with I, J and L, respectively. Applying the results of appendix A, we find the geometric conditions

$$\begin{aligned}
(de_+)_{\alpha\beta} &= -\frac{1}{2}(i_{I_1}\nabla_+\omega_1)_{\alpha\beta}\,, & (de_+)_{pq} &= -\frac{1}{2}(i_{I_2}\nabla_+\omega_2)_{pq} \\
(de_+)_{p\alpha} &= -2(i_{I_1}\nabla_+\omega_1)_{p\alpha}\,, & (de_+)_{\bar{p}\alpha} &= -2(i_{I_1}\nabla_+\omega_1)_{\bar{p}\alpha} \\
(de_+)_{ij}\,\omega_1^{ij} &= (\nabla_+\mathrm{Re}\chi_1)_{ij}\,\mathrm{Im}\chi_1^{ij}\,, & (\nabla_+\omega_1)_{pq} &= 0\,, \\
(de_+)_{ij}\,\omega_2^{ij} &= (\nabla_+\mathrm{Re}\chi_2)_{ij}\,\mathrm{Im}\chi_2^{ij}\,, & (\nabla_+\omega_2)_{\alpha\beta} &= 0\,,
\end{aligned} \tag{5.5}$$

where $\alpha, \beta = 1, 2$ and $p, q = 3, 4$. From these it is also straightforward to express the components of $H_-^{\mathfrak{k}\perp}$ in terms of the geometry. In particular, one has that

$$\begin{aligned}
H_{-\alpha\beta} &= -\frac{1}{2}(i_{I_1}\nabla_-\omega_1)_{\alpha\beta} = (\nabla_-\mathrm{Re}\chi_1)_{\alpha i}\,\mathrm{Re}(\chi_1)_\beta{}^i + (\nabla_-\mathrm{Im}\,\chi_1)_{\alpha i}\,\mathrm{Im}(\chi_1)_\beta{}^i \\
H_{-pq} &= -\frac{1}{2}(i_{I_2}\nabla_-\omega_2)_{pq} = (\nabla_-\mathrm{Re}\chi_2)_{pi}\,\mathrm{Re}(\chi_2)_q{}^i + (\nabla_-\mathrm{Im}\,\chi_2)_{pi}\,\mathrm{Im}(\chi_2)_q{}^i \\
H_{-p\alpha} &= -2(i_{I_1}\nabla_-\omega_1)_{p\alpha} = -2(i_{I_2}\nabla_-\omega_2)_{p\alpha} = 2(\nabla_-\mathrm{Re}\chi_1)_{pi}\,(\mathrm{Re}\,\chi_1)_\alpha{}^i \\
&= -2(\nabla_-\mathrm{Re}\chi_2)_{\alpha i}\,(\mathrm{Re}\,\chi_2)_p{}^i\,, \\
H_{-\bar{p}\alpha} &= -2(i_{I_1}\nabla_-\omega_1)_{\bar{p}\alpha} = -2(i_{I_2}\nabla_-\omega_2)_{\bar{p}\alpha} = 2(\nabla_-\mathrm{Re}\chi_1)_{\bar{p}i}\,(\mathrm{Re}\,\chi_1)_\alpha{}^i\,, \\
&= -2(\nabla_-\mathrm{Re}\chi_2)_{\alpha i}\,(\mathrm{Re}\,\chi_2)_{\bar{p}}{}^i\,, \\
H_{-ij}\,\omega_1^{ij} &= (\nabla_-\mathrm{Re}\chi_1)_{ij}\,\mathrm{Im}\chi_1^{ij}\,, \quad (\nabla_-\omega_1)_{pq} = 0\,, \\
H_{-ij}\,\omega_2^{ij} &= (\nabla_-\mathrm{Re}\chi_2)_{ij}\,\mathrm{Im}\chi_2^{ij}\,, \quad (\nabla_-\omega_2)_{\alpha\beta} = 0\,.
\end{aligned} \tag{5.6}$$

This concludes the analysis of the conditions along the light-cone directions.

Next consider the parallel transport equations along the transverse directions. It turns out that

$$\begin{aligned}
H^{\mathrm{rest}} &= -i_I\tilde{d}\omega_I - 2\mathcal{N}(I) = \star(\tilde{d}\omega_I \wedge \omega_I) - \frac{1}{2}\star(\theta_{\omega_I} \wedge \omega_I \wedge \omega_I) + \mathcal{N}(I) \\
&= -i_J\tilde{d}\omega_J - 2\mathcal{N}(J) = \star(\tilde{d}\omega_J \wedge \omega_J) - \frac{1}{2}\star(\theta_{\omega_J} \wedge \omega_J \wedge \omega_J) + \mathcal{N}(J) \\
&= -i_L\tilde{d}\omega_L - 2\mathcal{N}(L) = \star(\tilde{d}\omega_L \wedge \omega_L) - \frac{1}{2}\star(\theta_{\omega_L} \wedge \omega_L \wedge \omega_L) + \mathcal{N}(L)\,.
\end{aligned} \tag{5.7}$$

In addition, the W_2 Gray-Hervella classes for each endomorphism should also vanish, i.e.

$$W_2(I) = W_2(J) = W_2(L) = 0\,. \tag{5.8}$$

This in turn implies that the Nijenhuis tensors of all endomorphisms are skew-symmetric.

5.2 N=1

As in previous $N = 1$ cases the dilatino Killing spinor equation is

$$\mathcal{A}(1 + e_{1234}) = 0\,. \tag{5.9}$$

The solution has been given in either (3.8) or equivalently (3.9). It can be easily decomposed in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ representations but the way it is stated in (3.9) suffices for our purpose.

$\times^2 \mathrm{SU}(2) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	θ
$N=1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I), \mathcal{N}(J), \mathcal{N}(L) \neq 0$	$-$
$N=2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = 0, \mathcal{N}(J), \mathcal{N}(J) \neq 0$	$-$
$N=3$	$\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = \mathcal{N}(J) = 0, \mathcal{N}(L) \neq 0$	$\theta_{\omega_I} = \theta_{\omega_J}$
$N=4$	$(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = \mathcal{N}(J) = \mathcal{N}(L) = 0$	$\theta_{\omega_I} = \theta_{\omega_J} = \theta_{\omega_L}$

Table 5: The differences in the geometry of descendants are in the non-vanishing components of de^-, and $\mathcal{N}(I)$, $\mathcal{N}(J)$ and $\mathcal{N}(L)$, and the relation between the Lee forms. $-$ indicates that there is no relation between the Lee forms. It is understood that the remaining conditions of the dilatino Killing spinor equation of $N=1$ supersymmetric backgrounds are valid.

5.3 N=2

The dilatino Killing spinor equation is

$$\mathcal{A}1 = 0 \ . \tag{5.10}$$

The solution has been given in either (3.11) or equivalently (3.12). Again the solution can be easily decomposed in $\mathrm{SU}(2) \times \mathrm{SU}(2)$ representations but the way it has been expressed in (3.12) will suffice.

5.4 N=3

The dilatino Killing spinor equation is

$$\mathcal{A}1 = 0 \, , \quad \mathcal{A}(e_{12} - e_{34}) = 0 \ . \tag{5.11}$$

The solution of the dilatino Killing spinor equation is the same as for the $N=3$ supersymmetric backgrounds with $\mathrm{Sp}(2) \ltimes \mathbb{R}^8$-invariant parallel spinors. These conditions have already been stated in (4.8).

5.5 Comparison with N=4

The solution of the dilatino Killing spinor equation has been given in [28]. This is summarized as

$$\begin{aligned} \partial_+ \Phi &= 0 \, , \quad de^- \in (\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \oplus_s \mathbb{R}^8 \, , \\ \mathcal{N}(I)_{ijk} &= \mathcal{N}(J)_{ijk} = \mathcal{N}(L)_{ijk} = 0 \, , \\ 2\partial_i \Phi - H_{-+i} &= (\theta_{\omega_I})_i = (\theta_{\omega_J})_i = (\theta_{\omega_L})_i \ . \end{aligned} \tag{5.12}$$

The conditions that arise from the gravitino Killing spinor equation are the same in all cases. The differences in the geometry of the descendants that arise from the dilatino Killing spinor equation are summarized in table 5.

6. $\mathrm{SU}(2) \ltimes \mathbb{R}^8$ and its descendants

A basis in the space of $\hat{\nabla}$-parallel spinors $\mathcal{P}$ is

$$1 \, , \quad e_{12} \, , \quad e_{13} + e_{24} \ . \tag{6.1}$$

N	Stab_Σ
1	$\mathrm{SU}(2)\times\mathrm{SU}(2)$
2	$\mathrm{SU}(2)$

Table 6: The first column denotes the number of supersymmetries and the second column the stability subgroups of Killing spinors for $N \leq 2$ in $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1)\times\mathrm{Sp}(2)$.

It can be easily verified by a direct computation that the above spinors are invariant under $\mathrm{SU}(2)\ltimes\mathbb{R}^8$, where $\mathfrak{su}(2)$ is generated by

$$i(\Gamma^{1\bar{1}}-\Gamma^{2\bar{2}}-\Gamma^{3\bar{3}}+\Gamma^{4\bar{4}})\,,\quad i(\Gamma^{1\bar{2}}+\Gamma^{2\bar{1}}-\Gamma^{3\bar{4}}-\Gamma^{4\bar{3}})\,,\quad \Gamma^{1\bar{2}}-\Gamma^{2\bar{1}}+\Gamma^{3\bar{4}}-\Gamma^{4\bar{3}}\,. \tag{6.2}$$

Alternatively observe that chiral Majorana-Weyl representation of Spin(8), $\Delta^+_{\mathbf{8}}$, decomposes under $\mathrm{SU}(2)\times\mathrm{SU}(2)$ as $\Delta^+_{\mathbf{8}}=\oplus^4\mathbb{R}\oplus\mathbb{H}$, where the first four directions are spanned by the $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant spinors. Moreover $\mathrm{SU}(2)\times\mathrm{SU}(2)$ acts on $\mathbb{H}$ by left and right quaternionic multiplication. Consequently, the diagonal SU(2) subgroup leaves invariant an additional spinor, and so $\mathrm{SU}(2)\ltimes\mathbb{R}^8$ leaves invariant five spinors.

To investigate the descendants of $\mathrm{SU}(2)\ltimes\mathbb{R}^8$ first observe that $\Sigma(\mathcal{P})=\mathrm{Spin}(1,1)\times\mathrm{Sp}(2)$, where Sp(2) acts on $\mathcal{P}$ with the five-dimensional vector representation, $\mathrm{Sp}(2)=\mathrm{Spin}(5)$. This can be verified either by a direct computation or by observing that any three linearly independent spinors in $\Delta^+_{\mathbf{8}}$ have stability subgroup $\mathrm{Sp}(2)\subset\mathrm{Spin}(8)$. Again $\mathrm{Spin}(1,1)$ is generated by Γ^{+-}. The group Sp(2) acts transitively on the $S^4\subset\mathcal{P}$ with stability subgroup $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Using this, it is easy to construct all the descendants. The Stab_Σ groups are collected in table 6.

6.1 The geometry of the gravitino Killing spinor equation

The gravitino Killing spinor equation is

$$\hat{\nabla}1=\hat{\nabla}e_{12}=\hat{\nabla}(e_{13}+e_{24})=0\,, \tag{6.3}$$

which implies that $\mathrm{hol}(\hat{\nabla})\subseteq\mathrm{SU}(2)\ltimes\mathbb{R}^8$. In turn this is equivalent to requiring that the forms

$$e^-\,,\quad e^-\wedge\omega_I\,,\quad e^-\wedge\omega_J\,,\quad e^-\wedge\omega_L\,,\quad e^-\wedge\omega_Q \tag{6.4}$$

are $\hat{\nabla}$-parallel, where the first four forms are defined as in the $(\mathrm{SU}(2)\ltimes\mathrm{SU}(2))\ltimes\mathbb{R}^8$ case and

$$\omega_Q=e^1\wedge e^3+e^2\wedge e^4+e^{\bar{1}}\wedge e^{\bar{3}}+e^{\bar{2}}\wedge e^{\bar{4}}\,. \tag{6.5}$$

The form spinor bilinears are given in appendix D. The new endomorphism Q satisfies the algebraic conditions

$$IQ=-QI\,,\quad JQ=QJ\,,\quad QL=LQ\,,\quad Q^2=-\mathbf{1}_{8\times8}\,. \tag{6.6}$$

It is clear that Q is on the same footing as the other three. The $\mathfrak{su}(2)^\perp$ is spanned by those forms in $\Lambda^2(\mathbb{R}^8)$ which are (2,0) and (0,2) with respect to all endomorphisms. So the conditions on the geometry are four copies of those that we have found for $SU(4) \ltimes \mathbb{R}^8$. In particular, the gravitino Killing spinor equation along the directions transverse to the light-cone gives conditions like (5.7) and (5.8) but now for all endomorphisms I, J, L and Q, see also the general analysis of appendix A. The results are tabulated in table 7.

6.2 N=1

The dilatino Killing spinor equation is

$$\mathcal{A}(1 + e_{1234}) = 0 \tag{6.7}$$

The solution has been given in (3.9).

6.3 N=2

The dilatino Killing spinor equations are

$$\mathcal{A}1 = 0 \tag{6.8}$$

The solution has been given in (3.12).

6.4 N=3

The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}(e_{12} - e_{34}) = 0 \tag{6.9}$$

The solution has been given in (4.8).

6.5 N=4

The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}e_{12} = 0 \tag{6.10}$$

The solution has been given in (5.12).

6.6 N=5 and comparison with the descendants

The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}e_{12} = \mathcal{A}(e_{13} + e_{24}) = 0\ . \tag{6.11}$$

The solution to the dilatino Killing spinor equation is

$$\begin{aligned} \partial_+\Phi &= 0\,, \quad de^- \in \mathfrak{su}(2) \oplus_s \mathbb{R}^8\,, \\ \mathcal{N}(I)_{ijk} &= \mathcal{N}(J)_{ijk} = \mathcal{N}(L)_{ijk} = \mathcal{N}(Q)_{ijk} = 0\,, \\ 2\partial_i\Phi - H_{-+i} &= (\theta_{\omega_I})_i = (\theta_{\omega_J})_i = (\theta_{\omega_L})_i = (\theta_{\omega_Q})_i\ . \end{aligned} \tag{6.12}$$

The conditions that arise from the gravitino Killing spinor equation are the same in all cases. The $N = 5$ case and the descendants differ in the conditions that arise from the dilatino Killing spinor equation. The differences are summarized in table 7.

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$SU(2) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	θ
$N = 1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I), \mathcal{N}(J)$, $\mathcal{N}(L), \mathcal{N}(Q) \neq 0$	$-$
$N = 2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = 0$ $\mathcal{N}(J), \mathcal{N}(J), \mathcal{N}(Q) \neq 0$	$-$
$N = 3$	$\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = \mathcal{N}(J) = 0$, $\mathcal{N}(L), \mathcal{N}(Q) \neq 0$	$\theta_{\omega_I} = \theta_{\omega_J}$
$N = 4$	$(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = \mathcal{N}(J) = \mathcal{N}(L) = 0$ $\mathcal{N}(Q) \neq 0$	$\theta_{\omega_I} = \theta_{\omega_J} = \theta_{\omega_L}$
$N = 5$	$\mathfrak{su}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I) = \mathcal{N}(J) =$ $\mathcal{N}(L) = \mathcal{N}(Q) = 0$	$\theta_{\omega_I} = \theta_{\omega_J} = \theta_{\omega_L} = \theta_{\omega_Q}$

Table 7: As in previous cases, the differences in the geometry of descendants are in the non-vanishing components of de^-, and $\mathcal{N}(I)$, $\mathcal{N}(J)$, $\mathcal{N}(L)$, and $\mathcal{N}(Q)$ and the relation between the Lee forms. $-$ indicates that there is no relation between the Lee forms. It is understood that the remaining conditions of the dilatino Killing spinor equation of $N = 1$ supersymmetric backgrounds are valid.

7. $U(1) \ltimes \mathbb{R}^8$ and its descendants

A complex basis in the space of parallel spinors $\mathcal{P}$ is

$$1, \quad e_{12}, \quad e_{13} \; . \tag{7.1}$$

The presence of backgrounds with six parallel spinors is a direct consequence of the previous $SU(2) \ltimes \mathbb{R}^8$ case. To see this, we decompose $\Delta^+_{\mathbf{8}}$ under $SU(2)$ as $\Delta^+_{\mathbf{8}} = \oplus^5\mathbb{R} \oplus \mathbb{R}^3$, where the first five singlets span the five $SU(2) \ltimes \mathbb{R}^8$-invariant spinors. Since $SU(2)$ acts with the vector representation on $\mathbb{R}^3$, there is an additional invariant spinor with stability subgroup $U(1) \ltimes \mathbb{R}^8$. In the basis chosen above, $\mathfrak{u}(1)$ is generated by

$$i(\Gamma^{1\bar{1}} - \Gamma^{2\bar{2}} - \Gamma^{3\bar{3}} + \Gamma^{4\bar{4}}) \; . \tag{7.2}$$

To investigate the descendants of $U(1) \ltimes \mathbb{R}^8$ first observe that $\Sigma(\mathcal{P}) = Spin(1,1) \times SU(4)$, where $SU(4)$ acts on $\mathcal{P}$ with the real six-dimensional vector representation, $SU(4) = Spin(6)$. This can easily be seen from previous results by a direct commutation. Alternatively, it is a consequence of the fact that the stability subgroup in $Spin(8)$ of two linearly independent spinors in $\Delta^+_{\mathbf{8}}$ is $SU(4)$, and that $SU(4)$ acts on the remaining spinors with the six-dimensional representation. The descendants can be easily found using group theory and the observation that $SU(4)$ acts transitively on the S^5 in $\mathcal{P}$ with stability subgroup $Sp(2)$. The $Stab_\Sigma$ groups have been collected in table 8.

7.1 The geometry of the gravitino Killing spinor equation

The gravitino Killing spinor equation is

$$\hat{\nabla} 1 = \hat{\nabla} e_{12} = \hat{\nabla} e_{13} = 0 \; . \tag{7.3}$$

N	Stab_Σ
1	Sp(2)
2	SU(2) × SU(2)
3	SU(2)

Table 8: The first column denotes the number of supersymmetries and the second column the stability subgroups of Killing spinors for $N \leq 3$ in $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times \mathrm{SU}(4)$.

This is equivalent to requiring that $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{U}(1) \ltimes \mathbb{R}^8$. Explicitly, the solution is

$$\begin{aligned}&\hat{\Omega}_{A,B+} = \hat{\Omega}_{A,\alpha\beta} = \hat{\Omega}_{A,\alpha\bar{\beta}} = 0\,, \quad (\alpha \neq \beta)\,,\\ &\hat{\Omega}_{A,1\bar{1}} = -\hat{\Omega}_{A,2\bar{2}} = -\hat{\Omega}_{A,3\bar{3}} = \hat{\Omega}_{A,4\bar{4}}\,.\end{aligned} \tag{7.4}$$

The condition that $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{U}(1) \ltimes \mathbb{R}^8$ is also equivalent to requiring that the forms

$$e^-\,, \quad e^- \wedge \omega_I\,, \quad e^- \wedge \omega_J\,, \quad e^- \wedge \omega_L\,, \quad e^- \wedge \omega_Q\,, \quad e^- \wedge \omega_T\,, \tag{7.5}$$

are $\hat{\nabla}$-parallel, where the first five forms are defined as in the $\mathrm{SU}(2) \ltimes \mathbb{R}^8$ case and

$$\omega_T = -i(e^1 \wedge e^{\bar{1}} - e^2 \wedge e^{\bar{2}} + e^3 \wedge e^{\bar{3}} - e^4 \wedge e^{\bar{4}})\ . \tag{7.6}$$

The form spinor bilinears are given in appendix D. The new endomorphism obeys the algebraic conditions

$$IT = TI\,, \quad JT = TJ\,, \quad TL = LT\,, \quad TQ = -QT\,, \quad T^2 = -\mathbf{1}_{8\times 8}\ . \tag{7.7}$$

It is clear that the endomorphism T is on the same footing as the other four. So the conditions on the geometry are five copies of those that we have found for $\mathrm{SU}(4) \ltimes \mathbb{R}^8$ case. In particular, the gravitino Killing spinor equation along the directions transverse to the light-cone gives conditions like (5.7) and (5.8) but now for all endomorphisms I, J, L, Q and T, see also appendix A. The results are tabulated in table 9.

7.2 N=1

The dilatino Killing spinor equation is

$$\mathcal{A}(1 + e_{1234}) = 0\ . \tag{7.8}$$

The solution has been given in (3.9).

7.3 N=2

The dilatino Killing spinor equation is

$$\mathcal{A}1 = 0\ . \tag{7.9}$$

The solution has been given in (3.12).

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7.4 N=3

The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}(e_{12} + e_{34}) = 0 \ . \tag{7.10}$$

The solution has been given in (4.8).

7.5 N=4

The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}e_{12} = 0 \tag{7.11}$$

The solution has been given in (5.12).

7.6 N=5

The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}e_{12} = \mathcal{A}(e_{13} + e_{24}) = 0 \tag{7.12}$$

The solution has been given in (6.12).

7.7 N=6 and comparison with the descendants

The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}e_{12} = \mathcal{A}e_{13} = 0 \ . \tag{7.13}$$

The solution can be written as

$$\begin{aligned} \partial_+\Phi = 0\,, \quad de^- \in \mathfrak{u}(1) \oplus_s \mathbb{R}^8\,, \\ \mathcal{N}(I)_{ijk} = \mathcal{N}(J)_{ijk} = \mathcal{N}(L)_{ijk} = \mathcal{N}(Q)_{ijk} = \mathcal{N}(T)_{ijk} = 0\,, \\ 2\partial_i\Phi - H_{-+i} = (\theta_{\omega_I})_i = (\theta_{\omega_J})_i = (\theta_{\omega_L})_i = (\theta_{\omega_Q})_i = (\theta_{\omega_T})_i \ . \end{aligned} \tag{7.14}$$

Again, the conditions that arise from the gravitino Killing spinor equation are common to all cases. So the differences arise from the conditions implied by the dilatino Killing spinor equation. These have been summarized in table 9.

8. The descendants of $\mathbb{R}^8$

A complex basis in the space of $\hat{\nabla}$-parallel spinors $\mathcal{P}$ is

$$1\,, \quad e_{ij}\,, \quad i,j \leq 4. \tag{8.1}$$

It is clear that these spinors are invariant under $\mathbb{R}^8$. Direct inspection reveals that $\mathcal{P}$ can be identified with the positive chirality Majorana-Weyl representation $\Delta^+_{\mathbf{8}}$ of Spin(8), $\mathcal{P} = \Delta^+_{\mathbf{8}}$. Using this, we find that $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times \mathrm{Spin}(8)$, where the generator of $\mathrm{Spin}(1,1)$ is Γ^{+-}.

$\mathrm{U}(1) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	θ
$N=1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I), \mathcal{N}(J),$ $\mathcal{N}(L), \mathcal{N}(Q), \mathcal{N}(T) \neq 0$	$-$
$N=2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=0$ $\mathcal{N}(J), \mathcal{N}(J), \mathcal{N}(Q), \mathcal{N}(T) \neq 0$	$-$
$N=3$	$\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=0,$ $\mathcal{N}(L), \mathcal{N}(Q), \mathcal{N}(T) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}$
$N=4$	$(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=0$ $\mathcal{N}(Q), \mathcal{N}(T) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=\theta_{\omega_L}$
$N=5$	$\mathfrak{su}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=$ $\mathcal{N}(L)=\mathcal{N}(Q)=0, \mathcal{N}(T) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=$ $\theta_{\omega_L}=\theta_{\omega_Q}$
$N=6$	$\mathfrak{u}(1) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=$ $\mathcal{N}(L)=\mathcal{N}(Q)=\mathcal{N}(T)=0$	$\theta_{\omega_I}=\theta_{\omega_J}=$ $\theta_{\omega_L}=\theta_{\omega_Q}=\theta_{\omega_T}$

Table 9: As in previous cases, the differences in the geometry of descendants are in the non-vanishing components of de^-, and $\mathcal{N}(I)$, $\mathcal{N}(J)$, $\mathcal{N}(L)$, $\mathcal{N}(Q)$ and $\mathcal{N}(T)$, and the relation between the Lee forms. $-$ indicates that there is no relation between the Lee forms. It is understood that the remaining conditions of the dilatino Killing spinor equation of $N=1$ supersymmetric backgrounds are valid.

N	Stab_Σ
1	Spin(7)
2	SU(4)
3	Sp(2)
4	$\mathrm{SU}(2) \times \mathrm{SU}(2)$

Table 10: The first column denotes the number of supersymmetries and the second column the stability subgroups of Killing spinors for $N \leq 4$ in $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times \mathrm{Spin}(8)$.

In the investigation of the descendants with $N > 4$ it is also necessary to consider the normals to the parallel spinors. Using the definitions in section two, $\mathcal{Q}$ can also be identified with the positive chirality Majorana-Weyl representation $\Delta^+_{\mathbf{8}}$ of Spin(8), $\mathcal{Q} = \Delta^+_{\mathbf{8}}$. The identification of descendants of $\mathbb{R}^8$ is the most involved so far. Because of this, we shall describe each case in more detail. The descendants can be easily found using group theory and the observation that Spin(7) acts transitively on the S^7 in $\mathcal{P}$ with stability subgroup Spin(7). The Stab_Σ groups have been collected in table 10.

8.1 Geometry of the gravitino Killing spinor equation

The condition that $\mathrm{hol}(\hat{\nabla}) \subseteq \mathbb{R}^8$ is equivalent to requiring that the forms

$$e^- \,, \quad e^- \wedge e^i \,, \quad i = 1,2,3,4,6,7,8,9 \ . \tag{8.2}$$

are $\hat{\nabla}$-parallel. In this case, all the components of H are determined in terms of the geometry. To see this define the one-forms $(v_i) = \delta_{ij} e^j$. Then

$$H_{ijk} = -2\nabla_i (v_j)_k \,, \quad H_{-ij} = -2\nabla_- (v_i)_j \ . \tag{8.3}$$

In addition, one also has the geometric conditions

$$\nabla_i (v_j)_k = \nabla_{[i} (v_j)_{k]} \,, \quad (de_+)_{ij} = -2\nabla_+ (v_i)_j \ . \tag{8.4}$$

We can also describe the solution of the gravitino Killing spinor equation by choosing, $e^-, e^- \wedge \omega$, as $\hat{\nabla}$-parallel forms, where ω is a shorthand for a basis in the space of two-forms. This would have been more uniform with previous cases but the choice of the parallel forms in (8.2), even though they are not associated with spinor bilinears, leads to a simpler description of the spacetime geometry. As in previous cases, the dilatino Killing spinor equation imposes additional conditions on the fluxes and geometry.

8.2 N=1

As we have explained in section two, to choose the first Killing spinor in $\mathcal{P}$, it suffices to find the orbits of $\Sigma(\mathcal{P}) = \mathrm{Spin}(1,1) \times \mathrm{Spin}(8)$ in $\mathcal{P} = \Delta^+_{\mathbf{8}}$. There is only one type of orbit of co-dimension zero which has stability subgroup Spin(7) in $\Sigma(\mathcal{P})$. In particular, the dilatino Killing spinor equation is

$$\mathcal{A}(1 + e_{1234}) = 0 \ . \tag{8.5}$$

The solution of this equation expressed in Spin(7) representations is given in [28] and reads

$$\begin{aligned} \partial_+ \Phi = 0 \,, \quad de^- \in \mathfrak{spin}(7) \oplus_s \mathbb{R}^8 \,, \\ \partial_i \Phi - \frac{1}{2}(\theta_\phi)_i - \frac{1}{2} H_{-+i} = 0 \,, \end{aligned} \tag{8.6}$$

where $\theta_\phi = -\frac{1}{6} \star (\star \tilde{d}\phi \wedge \phi)$ is the Lee form of the Spin(7)-invariant form ϕ, and the Hodge dual has been taken with respect to the volume form $d\mathrm{vol} = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^6 \wedge e^7 \wedge e^8 \wedge e^9$.

8.3 N=2

The first Killing spinor ϵ is chosen as in the $N = 1$ case above, $\epsilon_1 = \epsilon$. To choose the direction of the second Killing spinor ϵ_2 observe that $\Delta^+_{\mathbf{8}}$ decomposes under the stability subgroup of the first normal as $\Delta^+_{\mathbf{8}} = \mathbb{R} < 1 + e_{1234} > \oplus \Lambda^1_{\mathbf{7}}(\mathbb{R}^7)$, i.e. $\mathcal{P}/\mathcal{K} = \Lambda^1_{\mathbf{7}}(\mathbb{R}^7)$. In addition, $\mathrm{Stab}(\mathcal{K}) = \mathrm{Spin}(1,1) \times \mathrm{Spin}(7)$, where Spin(7) is the stability subgroup of ϵ_1. Since Spin(7) acts transitively on the sphere in the space of one-forms of $\mathbb{R}^7$, $\mathrm{Stab}(\mathcal{P})$ has a single orbit in $\Lambda^1_{\mathbf{7}}(\mathbb{R}^7)$ of codimension zero with stability subgroup SU(4). So we can choose $\epsilon_2 = i(1 - e_{1234})$. The dilatino Killing spinor equation is

$$\mathcal{A}1 = 0 \ . \tag{8.7}$$

The solution organized in SU(4) representations is given in (3.12).

8.4 N=3

Next consider $\mathcal{K} = \mathbb{R} < \epsilon_1, \epsilon_2 >$, where e_1, ϵ_2 are the Killing spinors of the $N = 2$ case above, and observe that $\mathrm{Stab}(\mathcal{P}) = \mathrm{Spin}(1,1) \times SU(4) \times U(1)$. This group[8] is constructed from the stability subgroup SU(4) of both spinors, a U(1) generated by $i\Gamma^{1\bar{1}}$ which rotates ϵ_1 and ϵ_2 and a boost (scaling) generated by Γ^{+-}. Next observe that under $\mathrm{Stab}(\mathcal{K})$, $\mathcal{P}$ decomposes as $\mathcal{P} = \mathcal{K} \oplus \mathrm{Re}\Lambda^2_{\mathbf{6}}(\mathbb{C}^4)$, and so $\mathcal{P}/\mathcal{K} = \mathrm{Re}\Lambda^2_{\mathbf{6}}(\mathbb{C}^4)$. For this, we have used the decomposition $\Lambda^1_{\mathbf{7}}(\mathbb{R}^7) = \mathbb{R} < i(1 - e_{1234}) > \oplus \mathrm{Re}\Lambda^2_{\mathbf{6}}(\mathbb{C}^4)$ under SU(4), where $SU(4) = \mathrm{Spin}(6)$ acts with the vector representation[9] on $\mathrm{Re}\Lambda^2_{\mathbf{6}}(\mathbb{C}^4) = \mathbb{R}^6$. Thus $\mathrm{Spin}(1,1) \times SU(4) \times U(1)$ has one type of orbit in $\mathrm{Re}\Lambda^2_{\mathbf{6}}(\mathbb{C}^4)$ of codimension zero with stability subgroup $Sp(2) \times U(1)$, where $Sp(2) = \mathrm{Spin}(5)$. Thus a representative can be chosen as

$$\epsilon_3 = i(e_{12} + e_{34}) \ . \tag{8.8}$$

The dilatino Killing spinor equation is

$$\mathcal{A}1 = \mathcal{A}(e_{12} + e_{34}) = 0 \ . \tag{8.9}$$

The solution of this Killing spinor equation has been given in (4.8).

8.5 N=4

The $N = 4$ can be investigated in two ways. One is to use the gauge symmetry either to specify the Killing spinors or to determine their normals. It is the "self-dual" case under the correspondence

$$N \longleftrightarrow 8 - N \ . \tag{8.10}$$

The two ways of examining $N = 4$ are equivalent, so without loss of generality, we shall determine the Killing spinors. We begin by choosing the first three Killing spinors, $\epsilon_1, \epsilon_2, \epsilon_3$ as in the $N = 3$ case above. To determine the forth Killing spinor ϵ_4, let $\mathcal{K} = \mathbb{R} < \epsilon_1, \epsilon_2, \epsilon_3 >$ be the vector space spanned by the three Killing spinors. First observe that $\Sigma(\mathcal{K}) = \mathrm{Spin}(1,1) \times Sp(2) \times SU(2)$, where Sp(2) is the stability subgroup of the first three spinors and SU(2) acts on them with the vector representation. It suffices to focus on Sp(2). To determine $\mathcal{P}/\mathcal{K}$ recall the results from $N = 3$ and observe that $\mathrm{Re}\,\Lambda^2(\mathbb{C}^4) = \mathbb{R} < i(e_{12} + e_{34}) > \oplus \Lambda^1_{\mathbf{5}}(\mathbb{R}^5)$ under Sp(2), therefore $\mathcal{P}/\mathcal{K} = \Lambda^1_{\mathbf{5}}(\mathbb{R}^5)$. Moreover $Sp(2) = \mathrm{Spin}(5)$ acts with the vector representation on $\mathcal{P}/\mathcal{K}$ and so it has a unique type of orbit S^4 with stability subgroup $\mathrm{Spin}(4) = SU(2) \times SU(2)$. In fact $\mathrm{Stab}(\mathcal{K})$ has an orbit in $\mathcal{P}/\mathcal{K}$ of codimension zero, a representative can be chosen as

$$\epsilon_4 = i(e_{12} + e_{34}) \ . \tag{8.11}$$

[8]There may be discrete identifications in $\mathrm{Stab}(\mathcal{P})$ that we do not take into account because the analysis is focused on the Lie algebra level, i.e. one may have instead $\mathrm{Stab}(\mathcal{P}) = \mathrm{Spin}(1,1) \times (SU(4) \cdot U(1))$.

[9]The reality condition in $\Lambda^2(\mathbb{C}^4)$ is defined by the anti-linear map τ constructed from complex conjugation followed by a duality map. Observe that this commutes with the SU(4) action on $\Lambda^2(\mathbb{C}^4)$. So $\mathrm{Re}\Lambda^2(\mathbb{C}^4)$ is defined as the fixed point set of τ.

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N	Σ
1	$\mathrm{Spin}(1,1) \times \mathrm{Spin}(8)$
2	$\mathrm{Spin}(1,1) \times \mathrm{Spin}(7)$
3	$\mathrm{Spin}(1,1) \times SU(4) \times U(1)$
4	$\mathrm{Spin}(1,1) \times Sp(2) \times SU(2)$

Table 11: For $N > 4$ the same Σ groups are used to determine the normals of the Killing spinors.

The dilatino Killing spinor equation for $N = 4$ backgrounds becomes

$$\mathcal{A}1 = \mathcal{A}e_{12} = 0 \ . \tag{8.12}$$

The solution of this has been given in (5.12). In table 11, we summarize the groups Σ that have been used in the identification of the descendants.

8.6 N=5

The selection of Killing spinors for the remaining $N > 4$ backgrounds is straightforward from the analysis we have presented for the $N < 4$ cases and the correspondence $N \leftrightarrow 8 - N$. So the dilatino Killing spinor equations for $N > 4$ will be written down without further explanation. The choice of representatives is such that the Killing spinors of N-supersymmetric backgrounds are included in the $N+1$-supersymmetric ones.

The dilatino Killing spinor equation is

$$\mathcal{A}1 = \mathcal{A}e_{12} = \mathcal{A}(e_{13} + e_{24}) = 0 \ . \tag{8.13}$$

The solution has been given in (6.12).

8.7 N=6

The dilatino Killing spinor equation is

$$\mathcal{A}1 = \mathcal{A}e_{12} = \mathcal{A}e_{13} = 0 \ . \tag{8.14}$$

The solution has been given in (7.14).

8.8 N=7

The dilatino Killing spinor equation is

$$\mathcal{A}1 = \mathcal{A}e_{12} = \mathcal{A}e_{13} = \mathcal{A}(e_{23} - e_{14}) = 0 \ . \tag{8.15}$$

The solution is

$$\begin{aligned} &\partial_+\Phi = 0\,, \quad de^- \in \mathbb{R}^8\,, \\ &\mathcal{N}(I)_{ijk} = \mathcal{N}(J)_{ijk} = \mathcal{N}(L)_{ijk} = \mathcal{N}(Q)_{ijk} = \mathcal{N}(T)_{ijk} = \mathcal{N}(U)_{ijk} = 0\,, \\ &2\partial_i\Phi - H_{-+i} = (\theta_{\omega_I})_i = (\theta_{\omega_J})_i = (\theta_{\omega_L})_i = (\theta_{\omega_Q})_i = (\theta_{\omega_T})_i = (\theta_{\omega_U})_i\,, \end{aligned} \tag{8.16}$$

where the sixth endomorphism U is defined via the Hermitian form

$$\omega_U = e^1 \wedge e^4 + e^{\bar{1}} \wedge e^{\bar{4}} - e^2 \wedge e^3 - e^{\bar{2}} \wedge e^{\bar{3}} \,. \tag{8.17}$$

The new endomorphism satisfies the algebraic conditions

$$UI = -IU \,, \quad UJ = JU \,, \quad UL = LU \,, \quad UQ = -QU \,, \quad UT = TU, \quad U^2 = -\mathbf{1}_{8\times 8} \,. \tag{8.18}$$

The dilatino Killing spinor equations imply that all the Lee forms of the endomorphisms are equal. However, this does not imply that all components of $H^{\rm rest}$ vanish. In particular, the non-vanishing components are

$$\begin{aligned}
\tfrac{1}{2}H_{1\bar{2}\bar{3}} &= +H_{\bar{4}1\bar{1}} = -H_{\bar{4}2\bar{2}} = -H_{\bar{4}3\bar{3}} \,, \\
\tfrac{1}{2}H_{4\bar{2}\bar{3}} &= -H_{\bar{1}4\bar{4}} = H_{\bar{1}2\bar{2}} = H_{\bar{1}3\bar{3}} \,, \\
\tfrac{1}{2}H_{2\bar{1}\bar{4}} &= +H_{\bar{3}2\bar{2}} = -H_{\bar{3}1\bar{1}} = -H_{\bar{3}4\bar{4}} \,, \\
\tfrac{1}{2}H_{3\bar{1}\bar{4}} &= -H_{\bar{2}3\bar{3}} = H_{\bar{2}1\bar{1}} = H_{\bar{2}4\bar{4}} \;.
\end{aligned} \tag{8.19}$$

8.9 N=8 and comparison with the descendants

The solution of the dilatino Killing spinor equation of $N = 8$ supersymmetric backgrounds [28] is

$$\partial_+\Phi = 0 \,, \quad de^- \in \mathbb{R}^8 \,, \quad H_{ijk} = 0 \,, \quad 2\partial_i\Phi - H_{-+i} = 0 \;. \tag{8.20}$$

The conditions that arise from the gravitino Killing spinor equation are common in all cases. The differences arise from those of the dilatino Killing spinor equation. We have summarize these in table 12.

9. Descendants and reduction of holonomy

So far we have solved the Killing spinor equations for all supersymmetric backgrounds for which the stability subgroup of the parallel spinors is non-compact, $\mathrm{Stab}(\epsilon_1, \dots, \epsilon_L) = K \ltimes \mathbb{R}^8$. The question that arises is whether the Bianchi identity of H and the field equations impose additional conditions on the existence of the various descendants we have found. We shall show that[10] if

$$dH = 0 \,, \tag{9.1}$$

and the field equations are satisfied, then for the descendants $\mathrm{hol}(\hat{\nabla}) \subset K \ltimes \mathbb{R}^8$. So the holonomy of the $\hat{\nabla}$-connection is a *proper* subgroup of the stability group of the parallel spinors. Since the holonomy of $\hat{\nabla}$ reduces, the structure group of the spacetime may reduce as well. Alternatively, if

$$dH = 0 \,, \quad \mathrm{hol}(\hat{\nabla}) = K \ltimes \mathbb{R}^8 \tag{9.2}$$

[10] This assumption is sufficient. What is required is that the term involving dH in the appropriate Bianchi identity in appendix A does not contribute in the calculations for the parallel forms.

$SU(2) \ltimes \mathbb{R}^8$	de^-	$\mathcal{N}$	θ
$N=1$	$\mathfrak{spin}(7) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I), \mathcal{N}(J), \mathcal{N}(L)$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$-$
$N=2$	$\mathfrak{su}(4) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=0, \mathcal{N}(J), \mathcal{N}(J),$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$-$
$N=3$	$\mathfrak{sp}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=0, \mathcal{N}(L),$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}$
$N=4$	$(\mathfrak{su}(2) \oplus \mathfrak{su}(2)) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=0$ $\mathcal{N}(Q), \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=\theta_{\omega_L}$
$N=5$	$\mathfrak{su}(2) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=$ $\mathcal{N}(Q)=0, \mathcal{N}(T), \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=$ $\theta_{\omega_L}=\theta_{\omega_Q}$
$N=6$	$\mathfrak{u}(1) \oplus_s \mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=$ $\mathcal{N}(Q)=\mathcal{N}(T)=0, \mathcal{N}(U) \neq 0$	$\theta_{\omega_I}=\theta_{\omega_J}=$ $\theta_{\omega_L}=\theta_{\omega_Q}=\theta_{\omega_T}$
$N=7$	$\mathbb{R}^8$	$\mathcal{N}(I)=\mathcal{N}(J)=\mathcal{N}(L)=$ $\mathcal{N}(Q)=\mathcal{N}(T)=\mathcal{N}(U)=0$	$\theta_{\omega_I}=\theta_{\omega_J}=\theta_{\omega_L}=$ $\theta_{\omega_Q}=\theta_{\omega_T}=\theta_{\omega_U}$
$N=8$	$\mathbb{R}^8$	$H_{ijk}=0$	

Table 12: As in previous cases, the differences in the geometry of descendants are in the non-vanishing components of de^-, and $\mathcal{N}(I)$, $\mathcal{N}(J)$, $\mathcal{N}(L)$, $\mathcal{N}(Q)$, $\mathcal{N}(T)$ and $\mathcal{N}(U)$ and the relation between the Lee forms. In the $N=8$ case, $H^{\rm rest}=0$ and so all the Nijenhuis tensors and Lee forms vanish. $-$ indicates that there is no relation between the Lee forms. It is understood that the remaining conditions of the dilatino Killing spinor equation of $N=1$ supersymmetric backgrounds are valid.

and the field equations are satisfied, then the gravitino Killing spinor equations imply the dilatino ones, and all $\hat\nabla$-parallel spinors are Killing. So there are no descendants and the only backgrounds that exist are those investigated in [28].

To establish these, we shall investigate in detail the $\hat\nabla$-parallel forms on the spacetime that arise as a consequence of the gravitino Killing spinor equation, $dH=0$ and the field equations of type I backgrounds. We shall focus first on the $SU(4) \ltimes \mathbb{R}^8$ case. We shall find that the spacetime may admit more parallel forms than those that may have been expected from the $SU(4) \ltimes \mathbb{R}^8$ isotropy group of the Killing spinors alone. As a consequence, we shall show the two statements mentioned above.

9.1 Parallel forms of $SU(4) \ltimes \mathbb{R}^8$ backgrounds

Suppose that $dH=0$ and hol $\subseteq SU(4) \ltimes \mathbb{R}^8$. To find additional parallel forms, we use the integrability condition of the gravitino Killing spinor equation as well the Bianchi identities of the $\hat R$ curvature. These have been summarized in appendix A. Since we have assumed $dH=0$, the Bianchi identity gives

$$\hat R_{A[B,CD]} = -\frac{1}{3}\hat\nabla_A H_{BCD} \ . \tag{9.3}$$

To proceed, set $B=+, C=\alpha, D=\bar\beta$ in (9.3) and contract with $\delta^{\alpha\bar\beta}$. Using that

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$\mathrm{hol}(\hat\nabla) \subseteq SU(4) \ltimes \mathbb{R}^8$, e.g. $\hat R_{AB,\alpha}{}^{\alpha} = 0$, it is easy to see that (9.3) implies that

$$\tau_1 = iH_{+\alpha}{}^{\alpha}\, e^+ \tag{9.4}$$

is $\hat\nabla$-parallel. Therefore if $\tau_1 \neq 0$, then $\mathrm{hol}(\hat\nabla) \subset SU(4)$. However, if we insist that $\mathrm{hol}(\hat\nabla) = SU(4) \ltimes \mathbb{R}^8$, then $\tau_1 = 0$.

To continue, set $B = +, C = \alpha, D = \beta$ in (9.3) and use that $\mathrm{hol}(\hat\nabla) \subseteq SU(4) \ltimes \mathbb{R}^8$, i.e. $\hat R_{AB,+i} = \hat R_{AB,\alpha\beta} = 0$, then it is easy to show that the three-form

$$\tau_2 = \frac{1}{2} H_{+\alpha\beta}\, e^+ \wedge e^\alpha \wedge e^\beta \,, \tag{9.5}$$

is $\hat\nabla$-parallel,

$$\hat\nabla_A \tau_2 = 0 \,, \tag{9.6}$$

Since there is no such form invariant under $SU(4) \ltimes \mathbb{R}^8$, one can only conclude that either the holonomy of $\hat\nabla$ reduces to a proper subgroup of $SU(4) \ltimes \mathbb{R}^8$ or $\tau_2 = 0$.

Next set $B = \alpha, C = \beta, D = \gamma$ in (9.3), and use that $\mathrm{hol}(\hat\nabla) \subseteq SU(4) \ltimes \mathbb{R}^8$ and $\tau_2 = 0$, to show that

$$\tau_3 = \frac{1}{3!} H_{\alpha\beta\gamma} e^\alpha \wedge e^\beta \wedge e^\gamma \tag{9.7}$$

is also $\hat\nabla$-parallel, i.e.

$$\hat\nabla_A \tau_3 = 0 \ . \tag{9.8}$$

Again there is no such form invariant under $SU(4) \ltimes \mathbb{R}^8$. So either the holonomy of $\hat\nabla$ reduces to a proper subgroup of $SU(4) \ltimes \mathbb{R}^8$ or $\tau_3 = 0$. Insisting that $\mathrm{hol}(\hat\nabla) = SU(4) \ltimes \mathbb{R}^8$, we have to set $\tau_3 = 0$. The observation that the Nijenhuis tensor of a Riemannian manifolds with a $U(n)$-structure compatible with a connection with skew-symmetric torsion H, $dH = 0$, is $\hat\nabla$-parallel has been made in the context of supersymmetric sigma models in [38, 39, 9].

There are two additional parallel one-forms which can be found using the field equations

$$\hat R_{AC,}{}^{C}{}_{B} - 2\hat\nabla_A \partial_B \Phi = 0 \ . \tag{9.9}$$

Setting $B = +$ and using $\mathrm{hol}(\hat\nabla) \subseteq SU(4) \ltimes \mathbb{R}^8$, one can show that the one-form

$$\tau_4 = \partial_+ \Phi\, e^+ \,, \tag{9.10}$$

is $\hat\nabla$-parallel. Since there is no such one-form invariant under $SU(4) \ltimes \mathbb{R}^8$, either the holonomy of $\mathrm{hol}(\hat\nabla)$ reduces to a subgroup of $SU(4) \ltimes \mathbb{R}^8$ or $\tau_4 = 0$.

Next set $B = \alpha, C = \beta, D = \bar\gamma$ in (9.3), take the trace in $\beta, \bar\gamma$, and use $\tau_1 = \tau_2 = \tau_4 = 0$ and $\mathrm{hol}(\hat\nabla) \subseteq SU(4) \ltimes \mathbb{R}^8$ to find

$$\hat R_{A\beta,}{}^{\beta}{}_{\alpha} = -[\partial_A H_{\alpha\beta}{}^{\beta} - \hat\Omega_{A,}{}^{\delta}{}_{\alpha} H_{\delta\beta}{}^{\beta} - \hat\Omega_{A,}{}^{+}{}_{\beta} H_{\alpha+}{}^{\beta}] \tag{9.11}$$

Similarly set $B = +, C = -, D = \alpha$ in (9.3), to get that

$$\hat R_{A+,-\alpha} = -[\partial_A H_{+-\alpha} - \hat\Omega_{A,}{}^{\delta}{}_{\alpha} H_{+-\delta} - \hat\Omega_{A,}{}^{\bar\beta}{}_{-} H_{+\bar\beta\alpha}] \ . \tag{9.12}$$

Substituting these into the field equations

$$\hat{R}_{A\beta,}{}^{\beta}{}_{\alpha} + \hat{R}_{A+,-\alpha} - 2\hat{\nabla}_A \partial_\alpha \Phi = 0\,, \tag{9.13}$$

we find that the one-form

$$\tau_5 = (2\partial_i\Phi - \theta_i + H_{+-i})e^i\,, \tag{9.14}$$

is $\hat{\nabla}$-parallel. Again, since there is no such one-form invariant under $SU(4) \ltimes \mathbb{R}^8$, either the holonomy of $\mathrm{hol}(\hat{\nabla})$ reduces to a subgroup of $SU(4) \ltimes \mathbb{R}^8$ or $\tau_5 = 0$.

For backgrounds to have precisely $N = 1$ supersymmetry, neither τ_2 nor τ_3 should vanish. As a consequence of the analysis above, $\mathrm{hol}(\hat{\nabla}) \subset SU(4) \ltimes \mathbb{R}^8$ and so the holonomy reduces to a proper subgroup of the isotropy group of the parallel spinors.

Another consequence of the analysis above is that if $dH = 0$, the field equations are satisfied and $\mathrm{hol}(\hat{\nabla}) = SU(4) \ltimes \mathbb{R}^8$, then all $\hat{\nabla}$-parallel spinors are Killing. This is because in such a case $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$ which are precisely the conditions (3.12) that arise from the dilatino Killing spinor equation of $N = 2$ backgrounds.

9.2 Parallel forms and descendants

We shall now turn to show the two statements stated in the beginning of the section. We will treat all cases together apart from the $N = 7$ descendant of $\mathbb{R}^8$, which will be discussed separately. We begin by constructing the forms τ_1, τ_2 and τ_3 with respect all the endomorphisms I, J, L, and so on, available in each case.

If one of these is non-vanishing, and so a descendant exists, then the holonomy of $\hat{\nabla}$ reduces. This is because the invariant forms of non-compact isotropy groups $K \ltimes \mathbb{R}^8$ are of the type

$$e^- \wedge \psi \tag{9.15}$$

where ψ are forms in the "transverse" directions. Since τ_1, τ_2, τ_3 and τ_4 are not of this type, one concludes that either the holonomy reduces or they should vanish.

Assuming that $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ with respect to all endomorphisms, one can show, using the argument we have presented above to establish that τ_5 is parallel in the $SU(4) \ltimes \mathbb{R}^8$ case, that all the differences of Lee forms

$$\theta_{\omega_I} - \theta_{\omega_J}\,, \quad \theta_{\omega_I} - \theta_{\omega_L}\,, \tag{9.16}$$

and so on, are also $\hat{\nabla}$-parallel. Since again these forms are not invariant under $K \ltimes \mathbb{R}^8$, either they vanish or the holonomy of $\hat{\nabla}$ reduces to a subgroup of $K \ltimes \mathbb{R}^8$. If they do not vanish, the holonomy reduces and so we have established the first statement. If they do vanish, and so $\mathrm{hol}(\hat{\nabla}) = K \ltimes \mathbb{R}^8$, they imply the dilatino Killing spinor equations for all parallel spinors. This establishes the second statement.

One can allow the holonomy to be reduced. The pattern of reductions depends on the choice of parallel forms $\tau_1, \tau_2, \tau_3, \tau_5$ that will be allowed not to vanish. For example if $\tau_1 \neq 0$, but the rest are zero, then the holonomy reduces from $SU(4) \ltimes \mathbb{R}^8$ to $SU(4)$. Similarly if $\tau_1, \tau_3 \neq 0$ but the rest vanish, then the holonomy reduces to $SU(3)$ and so on.

N	Stab_Σ
1	$\mathbb{R}$

Table 13: The first column denotes the number of supersymmetries and the second column the stability subgroup of Killing spinor in $\Sigma(\mathcal{P}) = \mathrm{Spin}(2,1)$.

The pattern of reductions of holonomy in the other cases is more involved. For example, consider the $\mathrm{Sp}(2) \ltimes \mathbb{R}^8$ case. Suppose that $\tau_1 \neq 0$. Then the holonomy reduces to $\mathrm{Sp}(2)$. The holonomy can remain $\mathrm{Sp}(2)$ even if $\tau_2 \neq 0$. This is because one can take $\tau_2 = e^+ \wedge \omega_J$, where ω_J is the hermitian form of the J endomorphism associated with this case. Therefore if one allows appropriate reductions of the holonomy group, many descendants may exist.

Finally consider the $N = 7$ descendant of $\mathbb{R}^8$. The dilatino Killing spinor equations imply that $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau_5 = 0$. So it may appear that for this descendant, the holonomy does not reduce. However, this is not the case because there are additional parallel forms which are the non-vanishing components of H_{ijk}. In particular using (9.3), $\mathrm{hol}(\hat\nabla) \subseteq \mathbb{R}^8$ and $H_{+ij} = (de_+)_{ij} = 0$, it is easy to see that the three-form

$$H^{\mathrm{rest}} = \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k \,, \tag{9.17}$$

constructed from the components (8.19) is $\hat\nabla$-parallel. A direct inspection of the integrability condition reveals that if $\mathrm{hol}(\hat\nabla) = \mathbb{R}^8$, then $H^{\mathrm{rest}} = 0$ and there is supersymmetry enhancement to $N = 8$. If some components of (8.19) are non-vanishing the holonomy reduces, i.e. $\mathrm{hol}(\hat\nabla) \subset \mathbb{R}^8$. If it reduces to the identity the background preserves at least 8 supersymmetries. This arises as a consequence of the conditions $dH = \hat R = 0$ and the dilatino Killing spinor equation [32, 31]. The argument is also reviewed in section 13.

10. The descendants of G_2

A basis in the space of parallel spinors is

$$1 + e_{1234} \,, \quad e_{15} + e_{2345} \ . \tag{10.1}$$

Moreover $\Sigma(\mathcal{P}) = \mathrm{Spin}(2,1)$ which acts with the Majorana representation on $\mathcal{P}$. There is a single descendant background with $N = 1$ supersymmetry. The dilatino Killing spinor equation can be written as

$$\mathcal{A}(1 + e_{1234}) = 0 \ . \tag{10.2}$$

The stability subgroup of this spinor is given in table 13.

10.1 Geometry of the gravitino Killing spinor equation

The condition that the gravitino Killing spinor equation imposes on the geometry is that $\mathrm{hol}(\hat\nabla) \subseteq G_2$, and has been investigated in [28]. This is equivalent to requiring that the forms

$$e^+ \,, \quad e^- \,, \quad e^{\underline{1}} \,, \quad \varphi \,, \tag{10.3}$$

are $\hat{\nabla}$-parallel, where $\varphi = \mathrm{Re}\,[(e^2+ie^7)\wedge(e^3+ie^8)\wedge(e^4+ie^9)]-e^6\wedge(e^2\wedge e^7+e^3\wedge e^8+e^4\wedge e^9)$ is the G_2 invariant three-form. It is clear that in this case there are three $\hat{\nabla}$-parallel one-forms which we shall call collectively[11] e^a, $a = +,-,\underline{1}$. As we have already explained in appendix A, the associated vector fields e_a are Killing and $i_a H = \eta_{ab} de^b$.

The geometric condition that arises from the compatibility of e^a and φ conditions, see appendix A, is that

$$[(de^a)_{ij}]^{\mathbf{7}} = \frac{1}{6}\eta^{ab}\nabla_b\varphi_{mn[i}\varphi^{mn}{}_{j]}\,, \quad i,j,k,\dots = 2,3,4,6,7,8,9\ . \tag{10.4}$$

For this, we have used the decomposition $\Lambda^2(\mathbb{R}^7) = \Lambda^2_{\mathbf{7}} \oplus \Lambda^2_{\mathbf{14}}$, where $\Lambda^2_{\mathbf{14}} = \mathfrak{g}_2$. The remaining components of H are determined as

$$H^{\mathrm{rest}} = -\frac{1}{6}(\tilde{d}\varphi, \star\varphi)\varphi + \star\tilde{d}\varphi - \star(\theta\wedge\varphi)\,, \tag{10.5}$$

where

$$\theta = -\frac{1}{3}\star(\star\tilde{d}\varphi\wedge\varphi)\ . \tag{10.6}$$

Moreover, $\tilde{d}$ denotes the projection of the exterior derivative along the transverse directions and the $\star$ operation has been taken with volume form $d\mathrm{vol} = e^2\wedge e^3\wedge e^4\wedge e^6\wedge\cdots\wedge e^9$. The geometry of Riemannian seven-dimensional manifolds with G_2-structure [40] compatible with a connection with skew-symmetric torsion has been examined in detail in [9]. For use later, a straightforward computation reveals that

$$\theta_i = -\frac{1}{6}H_{kmn}\star\varphi^{kmn}{}_i\ . \tag{10.7}$$

In addition, one also finds the geometric (integrability) condition

$$\tilde{d}\star\varphi = -\theta\wedge\star\varphi\ . \tag{10.8}$$

This is equivalent to requiring that the G_2 class X_2 associated with the **14** representation vanishes, $X_2 = 0$. This is the only condition required for the existence of (10.5). This concludes the description of the geometry of the gravitino Killing spinor equation.

10.2 Geometry of $N = 1$ supersymmetric backgrounds

The solution of the dilatino equation is that which one derives for the $\mathrm{Spin}(7)\ltimes\mathbb{R}^8$ backgrounds [28]. Organizing the conditions in G_2 representations, one has

$$\begin{aligned}\partial_+\Phi = 0\,, \quad H_{+\underline{1}i} + \frac{1}{2}H_{+mn}\varphi^{mn}{}_i &= 0\,,\\ \partial_{\underline{1}}\Phi - \frac{1}{12}H_{ijk}\varphi^{ijk} - \frac{1}{2}H_{-+\underline{1}} &= 0\,,\\ \partial_i\Phi - \frac{1}{12}H_{jkm}\star\varphi^{jkm}{}_i - \frac{1}{4}H_{\underline{1}jk}\varphi^{jk}{}_i - \frac{1}{2}H_{-+i} &= 0\ .\end{aligned} \tag{10.9}$$

[11] We have underlined one direction to emphasize that it is real.

Using the relation between H and e^a established in appendix A, the above conditions can be rewritten as

$$
\begin{aligned}
\partial_+\Phi = 0\,, \quad [e_+, e_{\underline{1}}]_i - \frac{1}{2}(de_+)_{mn}\varphi^{mn}{}_i &= 0\,, \\
\partial_{\underline{1}}\Phi - \frac{1}{12}H_{ijk}\varphi^{ijk} - \frac{1}{2}H_{-+\underline{1}} &= 0\,, \\
\partial_i\Phi - \frac{1}{2}\theta_i - \frac{1}{4}(de_{\underline{1}})_{jk}\varphi^{jk}{}_i + \frac{1}{2}[e_-, e_+]_i &= 0\,.
\end{aligned} \tag{10.10}
$$

Note that $H_{-+\underline{1}}$ can also be written in terms of the $\hat{\nabla}$-parallel vector fields $e_+, e_-, e_{\underline{1}}$ as $H_{-+\underline{1}} = -g([e_-, e_+], e_{\underline{1}})$, see appendix A, but it is more convenient for simplicity of notation to leave it as it is in the equations.

There are various ways to interpret the above conditions. First observe that Φ is invariant only under the action of one of the three Killing vector fields. The second condition is expected from the $N = 1$ $\mathrm{Spin}(7) \ltimes \mathbb{R}^8$ results and decomposition of $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \Lambda^1_{\mathbf{7}}$ in $\mathfrak{g}_2$ representations. The third condition relates the singlet in the decomposition of H^{rest} in G_2 representations to the structure constants of $H_{-+\underline{1}}$ and the derivative of Φ along $e_{\underline{1}}$. Finally, the last condition can be thought of as a generalization of the conformal balanced condition. The additional terms involve the rotation of $e^{\underline{1}}$ and the commutator $[e_-, e_+]$.

Let $\mathfrak{h} = \mathbb{R} < e_-, e_+, e_{\underline{1}} >$. If $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, i.e. the algebra of three $\hat{\nabla}$-parallel vector fields closes, then the conditions (10.10) can be written as

$$
\begin{aligned}
\partial_+\Phi = 0\,, \quad (de_+)_{mn}\varphi^{mn}{}_i &= 0\,, \\
\partial_{\underline{1}}\Phi - \frac{1}{12}H_{ijk}\varphi^{ijk} - \frac{1}{2}H_{-+\underline{1}} &= 0\,, \\
\partial_i\Phi - \frac{1}{2}\theta_i - \frac{1}{4}(de_{\underline{1}})_{jk}\varphi^{jk}{}_i &= 0\,.
\end{aligned} \tag{10.11}
$$

In such a case, the spacetime is a principal bundle over a seven-dimensional base space, see [28] and appendix A. There are two cases to consider. If the isometry group is abelian, the curvature $\mathcal{F}^-$ of the principal bundle is a $\mathfrak{g}_2$ instanton, and $\mathcal{F}^+$ and $\mathcal{F}^{\underline{1}}$ take values in $\mathfrak{so}(7)$. Though in the two latter cases in (10.4), $(\mathcal{F}^+)_{\mathbf{7}}$ and $(\mathcal{F}^{\underline{1}})_{\mathbf{7}}$ are related to the covariant derivative of φ. It is clear from these that both the dilaton Φ and the three-form bilinear φ may depend on the coordinates of the fiber and so they are not functions of the base space only of the principal fibration. If the dilaton is invariant under $e_{\underline{1}}$, then the singlet in the decomposition of H vanishes.

A similar conclusion can also be reached in the case that the Lie algebra of isometries is $\mathfrak{sl}(2,\mathbb{R})$. One of the differences is that the singlet in the decomposition of H does not vanish even if the dilaton is invariant. In fact it is related to the structure constants of $\mathfrak{h}$ as it can be seen in the second equation in (10.11).

10.3 $N = 2$

The solution of the dilatino Killing spinor equation can be found[12] in [28]. It turns out

[12] In [28], the solution has been organized in this way only for the case that the algebra of isometries closes.

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that it can be written as

$$\begin{aligned}&\partial_a\Phi=0\,,\quad \epsilon_a{}^{bc}[e_b,e_c]_i-(de_a)_{mn}\varphi^{mn}{}_i=0\,,\quad \epsilon_{+-\underline{1}}=1\,,\\&\frac{1}{6}H_{ijk}\varphi^{ijk}+H_{-+\underline{1}}=0\,,\quad \partial_i\Phi-\frac{1}{2}\theta_i=0\,.\end{aligned}\tag{10.12}$$

The dilaton is invariant under all the three Killing vector fields. Moreover all $(de^a_{ij})_{\mathbf{7}}$ are related to the commutator $\epsilon_a{}^{bc}([X_b,X_c])_i$. In the case that the algebra of the three isometries closes, $\mathcal{F}^a$ takes values in $\mathfrak{g}_2$ and so the principal bundle connection is a $\mathfrak{g}_2$ instanton. The geometry has been investigated in detail in [28] and we shall not explain this further here.

The $N=1$ and $N=2$ differ. It is clear that the conditions that arise from the dilatino Killing spinor equations in the two cases are not the same. The main differences lie in the conditions on de^a_{ij} and whether the dilaton is invariant under the isometries of the backgrounds.

10.4 Reduction of holonomy

Reduction of the holonomy group can happen in backgrounds with both $N=1$ and $N=2$ supersymmetry. This is unlike the non-compact case where we have shown that the Bianchi identities and the field equations force a reduction of the holonomy only for the descendants. We shall again use the Bianchi identities to find the additional parallel forms on the spacetime.

To begin, suppose that $dH=0$. It has been shown in [28] that either $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$, where $\mathfrak{h}=\mathbb{R}<e_a>$, $a=-,+,\underline{1}$, or the holonomy of $\mathrm{hol}(\hat{\nabla})\subset G_2$. This is because[13] if $dH=0$, then the commutator of two $\hat{\nabla}$-parallel vector fields is $\hat{\nabla}$-parallel, see [28]. Thus either $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$ or there is an additional linearly independent vector field which is $\hat{\nabla}$-parallel and so $\mathrm{hol}(\hat{\nabla})\subset G_2$, i.e. the holonomy reduces. This can also be shown using the Bianchi identity (9.3), see also appendix A.

Applying the Bianchi identity (9.3) for $B=a,C=b,D=c$, we can show that H_{abc} are constant. In addition contracting the Bianchi identity (9.3) for $B=i,C=i,D=k$ with the φ, and using the condition that $\mathrm{hol}(\hat{\nabla})\subseteq G_2$, i.e.

$$\hat{R}_{AB,aD}=\hat{R}_{AB,ij}\varphi^{ij}{}_k=0\,,\tag{10.13}$$

one can also show that $H_{ijk}\varphi^{ijk}$ is constant as well.

Using (10.13) and the Bianchi identity (9.3) for $A=a,B=i,C=j$, one can show that the Lie-algebra valued one-form

$$\tau^a_1=\frac{1}{2}\,de^a_{ij}\,\varphi^{ij}{}_k\,e^k\,,\tag{10.14}$$

is $\hat{\nabla}$-parallel. Since τ^a_1 are linearly independent from e^a, either τ^a_1 vanishes or the holonomy of $\hat{\nabla}$ reduces to a subgroup of G_2. Observe that τ^a_1 is the 7-dimensional component of $\tilde{d}e^a$ in the decomposition of two-forms in G_2 representations.

[13] In fact a necessary is for H to be invariant.

Substituting $B = a$ in the field equations (9.9) and using $\mathrm{hol}(\hat{\nabla}) \subset G_2$, it is easy to see that

$$\tau_2 = \partial_a \Phi \, e^a \tag{10.15}$$

is $\hat{\nabla}$-parallel. Since e^a are $\hat{\nabla}$-parallel as well, this implies that $\partial_a \Phi = v_a$ are constant.

Next write the G_2 holonomy condition as

$$\frac{1}{2} \hat{R}_{AB,kl} \star \varphi^{kl}{}_{ij} = \hat{R}_{AB,ij} \ . \tag{10.16}$$

Setting $B = m$, contracting m and i, using the field equations (9.9) and the Bianchi identity (9.3), we find that

$$\tau_3 = (2\partial_i \Phi - \theta_i) e^i \tag{10.17}$$

is $\hat{\nabla}$-parallel. Since this one-form is linearly independent from e^a either $\tau_3 = 0$ or $\mathrm{hol}(\hat{\nabla}) \subset G_2$ and so the holonomy reduces.

First consider the consequences of the above $\hat{\nabla}$-parallel forms in the $N = 2$ backgrounds. If one insist that $\mathrm{hol}(\hat{\nabla}) = G_2$, then the field equations and $dH = 0$ imply all the conditions (10.12) that arise from the dilatino Killing spinor equation apart from $\partial_a \Phi = v_a = 0$ and $H_{ijk}\varphi^{ijk} = 0$ if $\mathfrak{h}$ is abelian, or $H_{ijk}\varphi^{ijk} + 6H_{-+\underline{1}} = 0$ if $\mathfrak{h}$ is non-abelian, respectively. (In the non-abelian case a simple argument implies that $v_a = 0$.) This is unlike the non-compact case, where under the same assumptions the gravitino Killing spinor equation implies all the conditions of the dilatino Killing spinor equation for the $N = L$ backgrounds.

Next consider the applications of the additional parallel forms in $N = 1$ backgrounds. If either $[\mathfrak{h}, \mathfrak{h}] \not\subseteq \mathfrak{h}$ and/or $\tau_1^a \neq 0$, then $\mathrm{hol}(\hat{\nabla}) \subset G_2$ and so the holonomy reduces. However, unlike the non-compact cases, there may be backgrounds with $N = 1$ supersymmetry and $\mathrm{hol}(\hat{\nabla}) = G_2$. For example take $\mathfrak{h}$ abelian, $\tau_1^a = 0$ and $12v_{\underline{1}} - H_{ijk}\varphi^{ijk} = 0$. This is a linear dilaton background.

11. The descendants of SU(3)

A complex basis in the space of parallel SU(3)-invariant spinors $\mathcal{P}$ is

$$1 \,, \quad e_{15} \ . \tag{11.1}$$

In this case $\Sigma(\mathcal{P}) = \mathrm{Spin}(3,1) \times \mathrm{U}(1)$, where $\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C})$ acts on $\mathcal{P}$ with the Majorana spinor representation and $\mathrm{U}(1)$ is generated by $\frac{i}{2}(\Gamma^{2\bar{2}} + \Gamma^{3\bar{3}})$. The generic orbit of $\Sigma(\mathcal{P})$ on $\mathcal{P}$ is of co-dimension one. To see this observe that the generic orbit of $\mathrm{Spin}(3,1)$ on $\mathcal{P}$ is of co-dimension two and so one can choose

$$\epsilon = \lambda_1(1 + e_{1234}) + i\lambda_2(1 - e_{1234}) \,, \quad \lambda_1^2 + \lambda_2^2 = 1 \ . \tag{11.2}$$

Moreover $\mathrm{U}(1)$ rotates the two spinors that appear in the expression above. So

$$\epsilon = \lambda_1(1 + e_{1234}) \ . \tag{11.3}$$

It is straightforward to choose the Killing spinors in all cases. For this observe that $\Sigma(\mathbb{R} < 1 + e_{1234} >) = (\mathrm{Spin}(1,1) \times \mathrm{U}(1)) \ltimes \mathbb{R}^2$. We simply state the results in the appropriate sections. The stability groups of the Killing spinors are summarized in table 14.

N	Stab_Σ
1	$\mathrm{U}(1) \ltimes \mathbb{R}^2$
2	$\mathbb{R}^2, \{1\}$

Table 14: The first column denotes the number of supersymmetries and the second column the stability subgroup of Killing spinor in $\Sigma(\mathcal{P}) = \mathrm{Spin}(3,1) \times \mathrm{U}(1)$.

11.1 Geometry of the gravitino Killing spinor equation

The gravitino Killing spinor equation implies that $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{SU}(3)$. This is equivalent to requiring [28] that the forms

$$\begin{aligned} e^a\,, \quad \omega &= \omega_I = -e^2 \wedge e^7 - e^2 \wedge e^8 - e^4 \wedge e^9\,, \\ \chi &= (e^2 + ie^7) \wedge (e^3 + ie^8) \wedge (e^4 + ie^9) \end{aligned} \tag{11.4}$$

are $\hat{\nabla}$-parallel, where $a = +, -, 1, \bar{1}$. So there are the four $\hat{\nabla}$-parallel vector fields and six transverse directions. In this case, $\mathfrak{k}^\perp$ is spanned by (2,0) and (0,2) forms with respect to I, and ω in $\Lambda^2(\mathbb{R}^6)$. As we have explained in appendix A, $H^{\mathfrak{k}^\perp}_{aij}$, $i,j = 2,3,4,7,8,9$, is determined both by de^a and the Levi-Civita covariant derivative of the remaining parallel forms. The compatibility between the different ways of expressing H leads to the geometric conditions

$$\begin{aligned} (de_a)^{2,0+0,2}_{ij} &= -\frac{1}{2}[i_I(\nabla_a\omega)]_{ij} \\ (de_a)_{ij}\omega^{ij} &= \frac{1}{6}(\nabla_a \mathrm{Re}\,\chi)_{k_1k_2k_3}\mathrm{Im}\chi^{k_1k_2k_3}\;. \end{aligned} \tag{11.5}$$

Moreover

$$H^{\mathrm{rest}} = -i_I\tilde{d}\omega - 2\mathcal{N} = \star\tilde{d}\omega - \star(\theta_\omega \wedge \omega) + \mathcal{N}\,, \tag{11.6}$$

where $\theta_\omega = -\star(\star\tilde{d}\omega \wedge \omega)$. One also finds the additional geometric constraints

$$W_2 = 0\,, \quad \theta_\omega = \theta_{\mathrm{Re}\chi}\,, \tag{11.7}$$

where again the vanishing of the Gray-Hervella class W_2 implies that the Nijenhuis tensor $\mathcal{N}$ is skew symmetric, and $\theta_{\mathrm{Re}\chi} = -\frac{1}{2}\star(\star\tilde{d}\mathrm{Re}\chi \wedge \mathrm{Re}\chi)$ is the Lee form of $\mathrm{Re}\chi$. The geometry of six-dimensional Riemannian manifolds with an SU(3)-structure [18] and compatible connection with skew-symmetric torsion have been extensively investigated, see [1, 2, 7, 8, 12, 13, 15 − 17, 20, 21]. The equality of the two Lee forms can also be expressed in SU(3) classes as $W_4 = W_5$.

It has been explained in [28] if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, where $\mathfrak{h} = \mathbb{R} < e_a >$, then $\mathfrak{h}$ is either abelian, $\mathbb{R} \oplus^3 \mathfrak{u}(1)$, $\mathbb{R} \oplus \mathfrak{su}(2)$, $\mathfrak{u}(1) \oplus \mathfrak{sl}(2,\mathbb{R})$, or a pp-wave algebra. This concludes the description of the geometry of the gravitino Killing spinor equation.

11.2 N=1

In this case the dilatino Killing spinor equation is

$$\mathcal{A}(1+e_{1234})=0\ . \tag{11.8}$$

The solution of the dilatino Killing spinor equation decomposed in SU(3) representations is

$$\begin{aligned}\partial_+\Phi=0\,,\quad H_{+1\bar{1}}+H_{+n}{}^n=0\,,\quad -H_{+\bar{1}\bar{n}}+\frac{1}{2}H_{+pq}\epsilon^{pq}{}_{\bar{n}}&=0\,,\\ \partial_{\bar{1}}\Phi-\frac{1}{6}H_{pqn}\epsilon^{pqn}-\frac{1}{2}H_{\bar{1}n}{}^n-\frac{1}{2}H_{-+\bar{1}}&=0\,,\\ \partial_{\bar{n}}\Phi+\frac{1}{2}H_{1pq}\epsilon^{pq}{}_{\bar{n}}-\frac{1}{2}H_{\bar{n}p}{}^p-\frac{1}{2}H_{\bar{n}1\bar{1}}-\frac{1}{2}H_{-+\bar{n}}&=0\,,\end{aligned} \tag{11.9}$$

where $p,q,n=2,3,4$. The dilaton is invariant under the e_+ isometry of the spacetime but not necessarily the rest. The remaining conditions can be interpreted in different ways. For example observe that the above condition implies that

$$de^-\in\mathfrak{spin}(7)\oplus_s\mathbb{R}^8\subset\mathfrak{so}(8)\oplus_s\mathbb{R}^8\ . \tag{11.10}$$

Alternatively, they can be seen as relating the structure constants and commutators of the Killing vector fields to the $\mathfrak{su}(3)^\perp$ components of de^-. In particular (11.9) can be rewritten as

$$\begin{aligned}\partial_+\Phi=0\,,\quad H_{+1\bar{1}}-\frac{i}{2}(de_+)_{ij}\omega^{ij}=0\,,\quad [e_+,e_{\bar{1}}]_{\bar{n}}+\frac{1}{2}(de_+)_{pq}\epsilon^{pq}{}_{\bar{n}}&=0\ .\\ \partial_{\bar{1}}\Phi-\frac{1}{24}\mathcal{N}_{pqn}\epsilon^{pqn}+\frac{i}{4}(de_{\bar{1}})_{ij}\omega^{ij}-\frac{1}{2}H_{-+\bar{1}}&=0\,,\\ \partial_{\bar{n}}\Phi+\frac{1}{2}(de_1)_{pq}\epsilon^{pq}{}_{\bar{n}}-\frac{1}{2}\theta_{\bar{n}}+\frac{1}{2}[e_1,e_{\bar{1}}]_{\bar{n}}+\frac{1}{2}[e_-,e_+]_{\bar{n}}&=0\ .\end{aligned} \tag{11.11}$$

If $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$, then one finds that

$$\begin{aligned}\partial_+\Phi=0\,,\quad H_{+1\bar{1}}-\frac{i}{2}(de_+)_{ij}\omega^{ij}=0\,,\quad (de_+)^{2,0}&=0\ .\\ \partial_{\bar{1}}\Phi-\frac{1}{24}\mathcal{N}_{pqn}\epsilon^{pqn}+\frac{i}{4}(de_{\bar{1}})_{ij}\omega^{ij}-\frac{1}{2}H_{-+\bar{1}}&=0\,,\\ \partial_{\bar{n}}\Phi+\frac{1}{2}(de_1)_{pq}\epsilon^{pq}{}_{\bar{n}}-\frac{1}{2}(\theta_\omega)_{\bar{n}}&=0\ .\end{aligned} \tag{11.12}$$

It is clear from this that although $\mathcal{L}_+\omega=0$ this is not the case for the rest of the parallel vector fields. In addition in all cases $\mathcal{L}_a\chi\neq0$, unless $\mathfrak{h}$ is abelian in which case $\mathcal{L}_+\chi=0$. This is in agreement with results in the maximal SU(3) case. Observe that even if one sets $\mathcal{N}=0$, the geometry of the Killing spinor equations is different from that of the $N=4$ case in [28].

11.3 N=2

The dilatino Killing spinor equation is either

$$\mathcal{A}1=0\,, \tag{11.13}$$

with $\mathrm{Stab}_\Sigma(1)=\mathbb{R}^2$ or

$$\mathcal{A}(1+e_{1234})=0\,,\quad \mathcal{A}(e_{15}+e_{2345})=0\ . \tag{11.14}$$

with $\mathrm{Stab}_\Sigma(1+e_{1234},e_{15}+e_{2345})=\{1\}$. So there are two cases to consider.

11.3.1 $\mathcal{A}1 = 0$

The solution of the dilatino Killing spinor equation $\mathcal{A}1 = 0$ is

$$\begin{aligned} &\partial_+\Phi = 0\,, \quad H_{+1\bar{1}} + H_{+n}{}^n = 0\,, \qquad H_{+1n} = H_{+mn} = H_{1mn} = H_{mpq} = 0\,, \\ &\partial_{\bar{1}}\Phi - \frac{1}{2}H_{\bar{1}n}{}^n - \frac{1}{2}H_{-+\bar{1}} = 0\,, \quad \partial_{\bar{n}}\Phi - \frac{1}{2}H_{\bar{n}p}{}^p - \frac{1}{2}H_{\bar{n}1\bar{1}} - \frac{1}{2}H_{-+\bar{n}} = 0\,, \end{aligned} \tag{11.15}$$

The dilaton is invariant only under the isometries generated by e_+. The above conditions imply that

$$\begin{aligned} &\partial_+\Phi = 0\,, \quad [e_+, e_1]_n = 0\,, \quad (de^-)^{2,0} = 0\,, \qquad (de^{\bar{1}})^{2,0} = 0\,, \\ &H_{+1\bar{1}} - \frac{i}{2}(de_+)_{ij}\,\omega^{ij} = 0\,, \qquad \mathcal{N}_{ijk} = 0\,, \quad \partial_{\bar{1}}\Phi + \frac{i}{4}(de_{\bar{1}})_{ij}\omega^{ij} - \frac{1}{2}H_{-+\bar{1}} = 0\,, \\ &\partial_{\bar{n}}\Phi - \frac{1}{2}(\theta_\omega)_{\bar{n}} + \frac{1}{2}[e_1, e_{\bar{1}}]_{\bar{n}} + \frac{1}{2}[e_-, e_+]_{\bar{n}} = 0\,. \end{aligned} \tag{11.16}$$

The difference between $N = 1$ and $N = 2$ is the vanishing of $\mathcal{N}$ and the restriction on $de^{\bar{1}}$ to be a (2,0)-form. Again $\mathcal{L}_a\chi \neq 0$. In particular, $W_1 = W_2 = 0$ for these backgrounds. If $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, the last condition is modified to

$$\partial_{\bar{n}}\Phi - \frac{1}{2}\theta_{\bar{n}} = 0\,. \tag{11.17}$$

11.3.2 $\mathcal{A}(1 + e_{1234}) = \mathcal{A}(e_{15} + e_{2345}) = 0$

The solution of the dilatino Killing spinor equation in this case is

$$\begin{aligned} &\partial_+\Phi = \partial_-\Phi = \partial_{\underline{1}}\Phi = 0\,, \quad H_{+1\bar{1}} + H_{+n}{}^n = 0\,, \quad H_{-1\bar{1}} - H_{-n}{}^n = 0\,, \\ &H_{-+\bar{1}} + H_{\bar{1}n}{}^n = -\frac{1}{6}H_{npq}\epsilon^{npq} - \frac{1}{6}H_{\bar{n}\bar{p}\bar{q}}\epsilon^{\bar{n}\bar{p}\bar{q}}\,, \\ &H_{+\bar{1}\bar{n}} = \frac{1}{2}H_{+pq}\epsilon^{pq}{}_{\bar{n}}\,, \quad H_{-\bar{1}\bar{n}} = -\frac{1}{2}H_{-pq}\epsilon^{pq}{}_{\bar{n}}\,, \\ &H_{-+\bar{n}} + H_{1\bar{1}\bar{n}} = \frac{1}{2}H_{1pq}\epsilon^{pq}{}_{\bar{n}} + \frac{1}{2}H_{\bar{1}pq}\epsilon^{pq}{}_{\bar{n}}\,, \\ &\partial_{\bar{1}}\Phi - \frac{1}{12}H_{npq}\epsilon^{npq} + \frac{1}{12}H_{\bar{n}\bar{p}\bar{q}}\epsilon^{\bar{n}\bar{p}\bar{q}} = 0\,, \\ &\partial_{\bar{n}}\Phi - \frac{1}{2}H_{\bar{n}p}{}^p - \frac{1}{4}H_{\bar{1}pq}\epsilon^{pq}{}_{\bar{n}} + \frac{1}{4}H_{1pq}\epsilon^{pq}{}_{\bar{n}} = 0\,. \end{aligned} \tag{11.18}$$

The conditions can be rewritten as

$$\begin{aligned} &\partial_+\Phi = \partial_-\Phi = \partial_{\underline{1}}\Phi = 0\,, \quad \frac{1}{3!}\epsilon_a{}^{bcd}H_{bcd} - \frac{1}{2}(de_a)_{ij}\omega^{ij} = 0\,, \quad a = +, - \\ &\frac{1}{3!}\epsilon_{\bar{1}}{}^{bcd}H_{bcd} - \frac{1}{2}(de_{\bar{1}})_{ij}\omega^{ij} = -\frac{\sqrt{2}}{6}\mathcal{N}_{ijk}\,\mathrm{Re}\chi^{ijk} \\ &[e_+, e_{\bar{1}}]_{\bar{n}} = -\frac{1}{2}(de_+)_{pq}\epsilon^{pq}{}_{\bar{n}}\,, \quad [e_-, e_{\bar{1}}]_{\bar{n}} = \frac{1}{2}(de_-)_{pq}\epsilon^{pq}{}_{\bar{n}}\,, \\ &[e_-, e_+]_{\bar{n}} + [e_1, e_{\bar{1}}]_{\bar{n}} = -\frac{1}{2}(de_1 + de_{\bar{1}})_{pq}\,\epsilon^{pq}{}_{\bar{n}}\,, \\ &\partial_6\Phi + \frac{1}{92}\mathcal{N}_{ijk}\,\mathrm{Im}\chi^{ijk} = 0\,, \quad \partial_{\bar{n}}\Phi - \frac{1}{2}(\theta_\omega)_{\bar{n}} + \frac{1}{4}(de_1 - de_{\bar{1}})_{pq}\,\epsilon^{pq}{}_{\bar{n}} = 0\,. \end{aligned} \tag{11.19}$$

The dilaton is invariant under the three out of four isometries of the background.

11.4 N=3

The dilatino Killing spinor equation is

$$\mathcal{A}(e_{15}+e_{2345})=0\,,\quad \mathcal{A}1=0\,. \tag{11.20}$$

The solution to the dilatino Killing spinor equations is

$$\begin{aligned}
\partial_+\Phi &= \partial_-\Phi=\partial_1\Phi=0\,, \quad H_{+1\bar{1}}+H_{+n}{}^n=0\,, \quad H_{-1\bar{1}}-H_{-n}{}^n=0\,, \quad H_{-+\bar{1}}+H_{\bar{1}n}{}^n=0\,,\\
H_{-\bar{1}\bar{n}} &= -\frac{1}{2}H_{-pq}\epsilon^{pq}{}_{\bar{n}}\,, \qquad H_{+1n}=H_{+pq}=0\,, \quad H_{pqn}=H_{1pq}=0\,,\\
H_{-+\bar{n}} &= \frac{1}{2}H_{\bar{1}pq}\epsilon^{pq}{}_{\bar{n}}-H_{\bar{n}1\bar{1}}\,, \qquad \partial_{\bar{n}}\Phi-\frac{1}{2}H_{\bar{n}p}{}^p-\frac{1}{4}H_{\bar{1}pq}\epsilon^{pq}{}_{\bar{n}}=0
\end{aligned} \tag{11.21}$$

These conditions can be rewritten as

$$\begin{aligned}
\partial_a\Phi &= 0\,, \quad \frac{1}{3!}\epsilon_a{}^{bcd}H_{bcd}-\frac{1}{2}(de_a)_{ij}\omega^{ij}=0\,, \quad (de^-)^{2,0}=(de^{\bar{1}})^{2,0}=0\,,\\
\mathcal{N}_{ijk} &= 0\,, \quad [e_-,e_{\bar{1}}]_{\bar{n}}=\frac{1}{2}(de_-)_{pq}\epsilon^{pq}{}_{\bar{n}}\,, \qquad [e_-,e_+]_{\bar{n}}+[e_1,e_{\bar{1}}]_{\bar{n}}=-\frac{1}{2}(de_{\bar{1}})_{pq}\epsilon^{pq}{}_{\bar{n}}\,,\\
[e_+,e_{\bar{1}}]_{\bar{n}} &= 0\,, \quad \partial_{\bar{n}}\Phi-\frac{1}{2}(\theta_\omega)_{\bar{n}}-\frac{1}{4}(de_{\bar{1}})_{pq}\epsilon^{pq}{}_{\bar{n}}=0\,.
\end{aligned} \tag{11.22}$$

Observe that if $(de^+)^{2,0}=(de^1)^{2,0}=0$, then the $N=3$ backgrounds admit an additional supersymmetry and so they admit four supersymmetries. We can show this by comparing the conditions above with those of $N=4$ backgrounds stated below. The same conclusion holds if $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$.

11.5 N=4

The solution to the dilatino Killing spinor equation given in [28] can be summarized as

$$\begin{aligned}
&\partial_a\Phi=0\,, \qquad \frac{1}{3!}\epsilon_a{}^{bcd}H_{bcd}-\frac{1}{2}H_{aij}\omega^{ij}=0\,, \qquad (de^a)^{2,0}=0\,,\\
&\mathcal{N}_{ijk}=0\,, \qquad \frac{1}{2}\epsilon_{ab}{}^{cd}H_{cdi}-H_{abj}I^j{}_i=0\,, \qquad \partial_i\Phi-\frac{1}{2}\theta_i=0\,.
\end{aligned} \tag{11.23}$$

In turn, these can be rewritten as

$$\begin{aligned}
&\partial_a\Phi=0\,, \qquad \frac{1}{3!}\epsilon_a{}^{bcd}H_{bcd}-\frac{1}{2}(de_a)_{ij}\omega^{ij}=0\,, \qquad (de^a)^{2,0}=0\,,\\
&\mathcal{N}_{ijk}=0\,, \qquad \frac{1}{2}\epsilon_{ab}{}^{cd}\,[e_c,e_d]_i-[e_a,e_b]_jI^j{}_i=0\,, \qquad \partial_i\Phi-\frac{1}{2}\theta_i=0\,.
\end{aligned} \tag{11.24}$$

The case that has been investigated in detail in [28] is that for which the algebra of four isometries closes. We shall not expand on this further here.

11.6 Reduction of holonomy

As in the G_2 case we have already investigated, we take that $dH=0$, $\mathrm{hol}(\hat{\nabla})\subseteq SU(3)$ and use the field equations to identify the additional $\hat{\nabla}$-parallel forms. As has been shown in [28], and elaborated on in the G_2 case, either $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$, $\mathfrak{h}=\mathbb{R}<e_a>$, or the holonomy of $\mathrm{hol}(\hat{\nabla})\subset SU(3)$. This is because the commutator of two parallel $\hat{\nabla}$-vectors is $\hat{\nabla}$-parallel.

Similarly, one can show that H_{abc} are constant and they can be identified with the structure constants of $\mathfrak{h}$.

Applying the Bianchi identity (9.3) for $B = a, C = p, D = \bar{q}$, contracting it with $\delta^{p\bar{q}}$ and using the condition that $\mathrm{hol}(\hat{\nabla}) \subseteq \mathrm{SU}(3)$, i.e.

$$\hat{R}_{AB,aC} = 0\,, \quad \hat{R}_{AB,p}{}^{p} = 0\,, \tag{11.25}$$

one can also show that

$$\tau_1 = iH_{aq}{}^{q}\, e^a \tag{11.26}$$

are $\hat{\nabla}$-parallel. Since e^a are also $\hat{\nabla}$-parallel, $iH_{aq}{}^{q} = u_a$ are constants. Similarly one can show that

$$\tau_2^a = \frac{1}{2} H^a{}_{pq} e^p \wedge e^q\,, \tag{11.27}$$

are also $\hat{\nabla}$-parallel. In this case, either $\tau_2^a = 0$ or $\mathrm{hol}(\hat{\nabla}) \subset \mathrm{SU}(3)$.

Next applying the Bianchi identity (9.3) for $B = p, C = q, D = n$, one can show that

$$\tau_3 = \frac{1}{3!}\mathcal{N}_{pqn}\, e^p \wedge e^q \wedge e^n = \frac{4}{3!} H_{pqn}\, e^p \wedge e^q \wedge e^n\,, \tag{11.28}$$

is $\hat{\nabla}$-parallel, see [38, 39, 9] for the properties of the Nijenhuis tensor of Riemannian almost complex manifolds with compatible $\hat{\nabla}$-connection. Since $\mathrm{Re}\chi$ and $\mathrm{Im}\chi$ are also (3,0) and (0,3) and $\hat{\nabla}$-parallel,

$$\tau_3 = \mathcal{N} = a\,\mathrm{Re}\chi + b\,\mathrm{Im}\chi\,, \tag{11.29}$$

for some constants $a, b \in \mathbb{R}$.

Using similar arguments to those we have made for the G_2 case, one can also show that

$$\tau_4 = \partial_a \Phi\, e^a \tag{11.30}$$

and

$$\tau_5 = (2\partial_i \Phi - (\theta_\omega)_i)\, e^i \tag{11.31}$$

are $\hat{\nabla}$-parallel. Since e^a are also $\hat{\nabla}$-parallel, $\partial_a \Phi = v_a$ are constants. Similarly, either $\tau_5 = 0$ or $\mathrm{hol}(\hat{\nabla}) \subset \mathrm{SU}(3)$.

The implication that these additional parallel forms have on the $N = L = 4$ supersymmetric backgrounds is as follows. It is clear that in this case the conditions $dH = 0$, $\mathrm{hol}(\hat{\nabla}) = \mathrm{SU}(3)$ and the field equations are not sufficient to imply the dilatino Killing spinor equations from the gravitino ones. For this to be the case, one has to impose in addition $\tau_3 = \tau_4 = 0$ and relate τ_1 to the structure constants of $\mathfrak{h}$.

The condition $\mathrm{hol}(\hat{\nabla}) = \mathrm{SU}(3)$ imposed on the $N = 3$ descendant implies enhancement of supersymmetry to $N = 4$. Backgrounds with $N = 3$ supersymmetry may exist but these require reduction of the holonomy. On the other hand backgrounds with $N = 1$ and $N = 2$ supersymmetry may exist even if $\mathrm{hol}(\hat{\nabla}) = \mathrm{SU}(3)$. For example, these can be linear dilaton backgrounds.

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N	Stab_Σ
1	$(\mathrm{SU}(2)\times\mathrm{SU}(2))\ltimes\mathbb{R}^4$
2	$(\mathrm{U}(1)\times\mathrm{SU}(2))\ltimes\mathbb{R}^4$, $\mathrm{U}(1)\times\mathrm{U}(1)$
3	$\mathrm{SU}(2)\ltimes\mathbb{R}^4$, $\mathrm{U}(1)$, $\{1\}$
4	$\mathrm{SU}(2)\ltimes\mathbb{R}^4$, $\mathrm{U}(1)$, $\{1\}$

Table 15: The first column denotes the number of supersymmetries and the second column the stability subgroup of Killing spinor in $\Sigma(\mathcal{P})=\mathrm{Spin}(5,1)\times\mathrm{SU}(2)$.

12. The descendants of SU(2)

A (complex) basis in the space of parallel spinors with $\mathrm{Stab}(\epsilon_1,\dots,\epsilon_8)=\mathrm{SU}(2)$ is

$$1\,,\quad e_{12}\,,\quad e_{15}\,,\quad e_{25}\ . \tag{12.1}$$

The subspace $\mathcal{P}$ in S^+ spanned by the above spinors can be identified with the positive chirality symplectic Majorana-Weyl representation $\Delta_{\mathbf{8}}^{+s}$ of $\mathrm{Spin}(5,1)$. To see this, first observe that $\mathrm{Stab}(\mathcal{P})=\mathrm{Spin}(5,1)\times\mathrm{Spin}(4)$, where the Lie algebra of $\mathrm{Spin}(5,1)$ is spanned by Clifford algebra directions $0,5,1,6,2,7$ and the Lie algebra of $\mathrm{Spin}(4)=\mathrm{SU}(2)\times\mathrm{SU}(2)$ is spanned by the Clifford algebra directions $3,4,8,9$. In addition we have $\mathrm{Stab}(\epsilon_1,\dots,\epsilon_8)=\mathrm{SU}(2)\subset\mathrm{Spin}(4)\subset\mathrm{Stab}(\mathcal{P})$, so $\Sigma(\mathcal{P})=\mathrm{Stab}(\mathcal{P})/\mathrm{Stab}(\epsilon_1,\dots,\epsilon_8)=\mathrm{Spin}(5,1)\times\mathrm{SU}(2)$. It is known that $\mathrm{Spin}(5,1)=\mathrm{SL}(2,\mathbb{H})$ and that $\mathrm{Spin}(5,1)$ does not admit Majorana-Weyl representations. However, it admits symplectic Majorana-Weyl representation after twisting with SU(2), i.e. taking two copies of the positive chirality (complex) Weyl representation and imposing a symplectic reality condition. This reality condition is precisely that inherited from the reality condition of the Majorana-Weyl spinors S^+ of $\mathrm{Spin}(9,1)$. In the explicit basis (12.1) of $\mathcal{P}$, one can show that the Lie algebra $\mathfrak{su}(2)$ of the SU(2) subgroup of $\Sigma(\mathcal{P})$ can be identified as $\mathfrak{su}(2)=\mathbb{R}<\Gamma^{34},\Gamma^{\bar 3\bar 4},\frac{i}{2}(\Gamma^{3\bar 3}+\Gamma^{4\bar 4})>$.

One can similarly examine $\mathcal{Q}$ which is required for investigating the normal spinors to the Killing spinors. In particular, one can show that $\mathcal{Q}=\Delta_{\mathbf{8}}^{-s}$, where $\Delta_{\mathbf{8}}^{-s}$ is the negative chirality Majorana-Weyl symplectic representation of $\mathrm{Spin}(5,1)$. Furthermore, $\Sigma(\mathcal{Q})=\mathrm{Spin}(5,1)\times\mathrm{SU}(2)$. The $N=8$ supersymmetric backgrounds have already been investigated in [28]. These are the backgrounds for which all parallel spinors are Killing. So it remains to investigate the backgrounds with $N<8$. For $4<N<8$, we shall use $\Sigma(\mathcal{Q})$ to choose directions for the normal spinors while for $1\leq N\leq 4$, we shall use $\Sigma(\mathcal{P})$ to choose directions in the space of parallel spinors. We shall not elaborate on the choice of normals to the Killing spinors for $N>4$ because it follows directly from the choice of Killing spinors for $N\leq 4$. So we shall simply state the dilatino Killing spinor equations in each case. The stability groups of the Killing spinors are summarized in table 15.

12.1 Geometry of the gravitino Killing spinor equation

The condition that $\mathrm{hol}(\hat\nabla)\subseteq\mathrm{SU}(2)$ is equivalent to requiring that the forms

$$e^a\,,\quad \omega_I=-(e^3\wedge e^8+e^4\wedge e^9)\,,\quad \omega_J+i\omega_K=(e^3+ie^8)\wedge(e^4+ie^9) \tag{12.2}$$

are $\hat{\nabla}$-parallel, where $a = +, -, 1, \bar{1}, 2, \bar{2}$. As in previous cases, $i_a H = (de_a)$ and in addition one has that

$$\begin{aligned} (de_a)^{2,0+0,2}_{ij} &= -\frac{1}{2}[i_I(\nabla_a \omega)]_{ij} \,, \\ (de_a)_{ij}\, \omega_I^{ij} &= (\nabla_a \omega_J)_{ij}\, \omega_K^{ij} \,, \end{aligned} \tag{12.3}$$

where $i, j = 3, 4, 8, 9$. Furthermore, one finds that

$$\begin{aligned} H^{\text{rest}} &= -i_I \tilde{d}\omega_I = -i_J \tilde{d}\omega_J \,, \\ \mathcal{N}(I)_{ijk} &= \mathcal{N}(J)_{ijk} = 0 \,. \end{aligned} \tag{12.4}$$

The geometry of Riemannian four-dimensional manifolds with an SU(2)- structure and compatible connection with skew-symmetric torsion has been extensively investigated, see [22, 1, 24, 5]. As in previous cases, we define $\mathfrak{h} = \mathbb{R} < e_a >$, and for the descendants $[\mathfrak{h}, \mathfrak{h}] \not\subseteq \mathfrak{h}$. However if we demand $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $\mathfrak{h}$ is a Lorentzian (5+1)-dimensional Lie algebra. These have been classified and the have been found to be

$$\begin{aligned} &\mathbb{R} \oplus^5 \mathfrak{u}(1)\,, \quad \mathbb{R} \oplus^2 \mathfrak{u}(1) \oplus \mathfrak{su}(2)\,, \quad \mathfrak{sl}(2, \mathbb{R}) \oplus^3 \mathfrak{u}(1)\,, \\ &\mathfrak{sl}(2, \mathbb{R}) \oplus^3 \mathfrak{su}(2)\,, \quad \mathfrak{cw}_4 \oplus^2 \mathfrak{u}(1)\,, \quad \mathfrak{cw}_6\,, \end{aligned} \tag{12.5}$$

where $\mathfrak{cw}_n$ denote pp-wave algebras of dimension $(n-1)+1$. For $N=8$ backgrounds, the dilatino Killing spinor equation implies that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ and $\mathfrak{h}$ is self-dual. These have been shown in [45] to be isomorphic to $\mathbb{R} \oplus^5 \mathfrak{u}(1)$, $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{su}(2)$ and $\mathfrak{cw}_6$.

12.2 N=1

The group action of $\Sigma(\mathcal{P}) = \text{Spin}(5,1) \times \text{SU}(2)$ on $\mathcal{P}$ can be most easily described in terms of quaternions. First identify $\mathcal{P} = \mathbb{H}^2$. Then $\text{Spin}(5,1) = \text{SL}(2, \mathbb{H})$ acts on $\mathcal{P}$ from the left with quaternionic matrix multiplication while SU(2) acts with quaternionic multiplication from the right, i.e.

$$\underline{x} \longrightarrow L\, \underline{x}\, \bar{a}\,, \quad \underline{x} \in \mathcal{P}\,, \quad L \in \text{SL}(2, \mathbb{H})\,, \quad a \in \text{SU}(2) = \text{Sp}(1) \tag{12.6}$$

where $\bar{a}$ is the quaternionic conjugate of a. It is easy then to see that $\Sigma(\mathcal{P})$ has a single orbit in $\mathcal{P}$ of codimension zero. The stability subgroup is $(\text{SU}(2)_L \times \text{SU}(2)_R) \ltimes \mathbb{R}^4$. So a representative can be chosen as $1 + e_{1234}$. In turn, the dilatino Killing spinor equation is

$$\mathcal{A}(1 + e_{1234}) = 0 \,. \tag{12.7}$$

The solution has been given in (3.9), and the conditions can be interpreted in a similar way. A different way of organizing the conditions is in terms of SU(2) representations. This allows to compare the results with the $N > 1$ cases. In particular, we find that

$$\begin{aligned} \partial_+ \Phi = 0\,, \quad H_{+1\bar{1}} + H_{+2\bar{2}} + (de_+)_n{}^n = 0\,, \quad - H_{+\bar{1}\bar{2}} + \frac{1}{2}(de_+)_{mn}\epsilon^{mn} &= 0\,, \\ [e_+, e_{\bar{1}}]_{\bar{n}} - [e_+, e_2]_m \epsilon^m{}_{\bar{n}} &= 0\,, \\ \partial_{\bar{1}}\Phi - \frac{1}{2}(de_2)_{mn}\epsilon^{mn} - \frac{1}{2}(de_{\bar{1}})_n{}^n - \frac{1}{2}H_{\bar{1}2\bar{2}} - \frac{1}{2}H_{-+\bar{1}} &= 0\,, \\ \partial_{\bar{2}}\Phi + \frac{1}{2}(de_1)_{mn}\epsilon^{mn} - \frac{1}{2}(de_{\bar{2}})_m{}^m - \frac{1}{2}H_{\bar{2}1\bar{1}} - \frac{1}{2}H_{-+\bar{2}} &= 0\,, \\ \partial_{\bar{n}}\Phi - [e_1, e_2]_m \epsilon^m{}_{\bar{n}} + \frac{1}{2}[e_2, e_{\bar{2}}]_{\bar{n}} + \frac{1}{2}[e_1, e_{\bar{1}}]_{\bar{n}} + \frac{1}{2}[e_-, e_+]_{\bar{n}} - \frac{1}{2}(\theta_{\omega_I})_{\bar{n}} &= 0\,, \end{aligned} \tag{12.8}$$

where $m, n = 3, 4$ are Hermitian indices. It is clear that even if we take $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $\mathfrak{h}$ is not necessarily a self-dual Lie algebra. In fact there is not a condition on the structure constants of $\mathfrak{h}$. The rotation $\tilde{d}e^a$ of the $\hat{\nabla}$-vector fields is also not restricted an a priori to lie in some subalgebra of $\mathfrak{so}(4)$. The dilaton is invariant only under e_+.

12.3 N=2

12.3.1 Killing spinors

The first Killing spinor ϵ_1 can be chosen as in the $N = 1$ case above, $\epsilon_1 = \epsilon = 1 + e_{1234}$. To continue, we shall first explain how the $\mathrm{Stab}(\epsilon) = (\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R) \ltimes \mathbb{R}^4$ acts on $\mathcal{P}/\mathcal{K}$, where $\mathcal{K} = \mathbb{R} < \epsilon >$. First identify $\mathcal{K}$ with the real axis in one of the quaternionic subspaces of $\mathcal{P} = \mathbb{H}^2$ and write $\mathcal{P} = \mathbb{R} < 1 + e_{1234} > \oplus \mathrm{Im}\mathbb{H} \oplus \mathbb{H}$. Thus we can set $\mathcal{P}/\mathcal{K} = \mathrm{Im}\mathbb{H} \oplus \mathbb{H}$. Then $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ acts as

$$(\mathbf{x}, y) \to (a\mathbf{x}\bar{a}, by\bar{a}) , \quad \mathbf{x} \in \mathrm{Im}\mathbb{H} , \quad y \in \mathbb{H} , \quad a \in \mathrm{SU}(2)_L , \quad b \in \mathrm{SU}(2)_R . \tag{12.9}$$

In addition the $\mathbb{R}^4$ subgroup acts with null boosts on $\mathcal{P}$ with fixed point set $\mathbb{R} < 1 + e_{1234} > \oplus \mathrm{Im}\mathbb{H}$. In the explicit basis for $\mathcal{P}$ in (12.1), $\mathfrak{su}(2)_L = \mathbb{R} < \Gamma^{\bar{1}\bar{2}} + \Gamma^{34}, \Gamma^{12} + \Gamma^{\bar{3}\bar{4}}, \frac{i}{2}(\Gamma^{1\bar{1}} + \Gamma^{2\bar{2}} - \Gamma^{3\bar{3}} - \Gamma^{4\bar{4}}) >$, $\mathfrak{su}(2)_R = \mathbb{R} < \Gamma^{1\bar{2}}, \Gamma^{\bar{1}2}, \frac{i}{2}(\Gamma^{1\bar{1}} - \Gamma^{2\bar{2}}) >$ and $\mathbb{R}^4 = \mathbb{R} < \Gamma^{-1}, \Gamma^{-\bar{1}}, \Gamma^{-2}, \Gamma^{-\bar{2}} >$. In addition, $\mathrm{Im}\mathbb{H} = \mathbb{R} < i(1 - e_{1234}), (e_{12} - e_{34}), i(e_{12} + e_{34}) >$ and $\mathbb{H}$ is spanned by the rest of the basis.

To continue first observe that $\Sigma(\mathcal{K}) = (\mathrm{Spin}(1,1) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R) \ltimes \mathbb{R}^4$, where $\mathrm{Spin}(1,1)$ is generated by Γ^{-+}. $\Sigma(\mathcal{K})$ has two types of orbits on $\mathcal{P}/\mathcal{K}$. One has codimension zero in $\mathrm{Im}\mathbb{H}$ and the other has codimension zero in $\mathcal{P}/\mathcal{K}$. To see this, consider the orbits of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ in $\mathrm{Im}\mathbb{H} \oplus \mathbb{H}$. There are three type of orbits. One orbit is an S^2 contained in $\mathrm{Im}\mathbb{H}$ with stability subgroup $\mathrm{U}(1)_L \times \mathrm{SU}(2)_R$, another is an S^3 contained in $\mathbb{H}$ with stability subgroup $(\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)/\mathrm{SU}(2) = \mathrm{SU}(2)$ and the third is a codimension two $(\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R)/\mathrm{U}(1)$ orbit in $\mathrm{Im}\,\mathbb{H} \oplus \mathbb{H}$. The latter orbit has representatives which have non-vanishing components in both $\mathrm{Im}\,\mathbb{H}$ and $\mathbb{H}$ subspaces. However one can show that such a representative lies in the same orbit of $\Sigma(\mathcal{K})$ as that of the S^3 using an $\mathbb{R}^4$ transformation. This can be easily seen by choosing the representative of the third orbit as

$$i\lambda_1(1 - e_{1234}) + \lambda_2(e_{15} + e_{2345}) , \quad \lambda_1, \lambda_2 \neq 0 . \tag{12.10}$$

Clearly an $\mathbb{R}^4$ transformation along the Γ^{-6} direction will transform a representative along $e_{15} + e_{2345}$ to the representative above. Thus $\Sigma(\mathcal{K})$ has only two orbits, one with stability subgroup $(\mathrm{U}(1)_L \times \mathrm{SU}(2)_R) \ltimes \mathbb{R}^4$ and the other stability subgroup $\mathrm{SU}(2)$ in $\Sigma(\mathcal{K})$. Therefore there are two choices for the second normal spinor each associated with the two orbits. Thus we can choose either

$$\epsilon_2 = i(1 - e_{1234}) \tag{12.11}$$

which lies in $\mathrm{Im}\mathbb{H}$, or

$$\epsilon_2 = (e_{15} + e_{1235}) \tag{12.12}$$

which lies in $\mathbb{H}$. So there are two dilatino Killing spinor equations to consider.

12.3.2 $\mathcal{A}1 = 0$

The solution of this Killing spinor equation has been given in (3.12). Decomposing the solution in SU(2) representations as in the $N = 1$ case, one finds,

$$\begin{aligned}
\partial_+\Phi = 0\,, \quad H_{+12} = 0\,, \qquad & [e_+, e_1]_n = [e_+, e_2]_n = [e_1, e_2]_n = 0\,, \\
(de_+)_{mn} = (de_1)_{mn} = (de_2)_{mn} = 0\,, \qquad & H_{+1\bar{1}} + H_{+2\bar{2}} + (de_+)_n{}^n = 0\,. \\
\partial_{\bar{1}}\Phi - \frac{1}{2}(de_{\bar{1}})_n{}^n - \frac{1}{2}H_{\bar{1}2\bar{2}} - \frac{1}{2}H_{-+\bar{1}} = 0\,, \quad & \partial_{\bar{2}}\Phi - \frac{1}{2}(de_{\bar{2}})_p{}^p - \frac{1}{2}H_{\bar{2}1\bar{1}} - \frac{1}{2}H_{-+\bar{2}} = 0\,, \\
\partial_{\bar{n}}\Phi + \frac{1}{2}[e_2, e_{\bar{2}}]_{\bar{n}} + \frac{1}{2}[e_1, e_{\bar{1}}]_{\bar{n}} + & \frac{1}{2}[e_-, e_+]_{\bar{n}} - \frac{1}{2}(\theta_{\omega_I})_{\bar{n}} = 0.
\end{aligned} \tag{12.13}$$

In general $[\mathfrak{h}, \mathfrak{h}] \not\subseteq \mathfrak{h}$. Moreover $\tilde{de}^- \in \mathfrak{u}(2)$ and $\tilde{de}^{\bar{1}}, \tilde{de}^{\bar{2}} \in \mathfrak{u}(2) \oplus_s \Lambda^{0,2}(\mathbb{C}^2)$ and $\tilde{de}^+ \in \mathfrak{so}(4)$. The rotations can be restricted further if for example $\mathfrak{h}$ is abelian. The dilaton is invariant only under e_+.

12.3.3 $\mathcal{A}(1 + e_{1234}) = \mathcal{A}(e_{15} + e_{2345}) = 0$

The solution of the latter is given in [28], see also (10.12). Expressing it in SU(2) representations, one finds

$$\begin{aligned}
\partial_+\Phi = \partial_-\Phi = \partial_{\underline{1}}\Phi &= 0\,, \\
H_{+1\bar{1}} + H_{+2\bar{2}} + (de_+)_n{}^n &= 0\,, \\
H_{-1\bar{1}} - H_{+2\bar{2}} - (de_-)_n{}^n &= 0\,, \\
H_{-+\bar{1}} + H_{\bar{1}2\bar{2}} + (de_{\bar{1}})_n{}^n &= -\frac{1}{2}(de_2)_{pq}\epsilon^{pq} - \frac{1}{2}(de_{\bar{2}})_{\bar{p}\bar{q}}\epsilon^{\bar{p}\bar{q}}\,, \\
H_{+\bar{1}\bar{2}} &= \frac{1}{2}(de_+)_{pq}\epsilon^{pq}\,, \\
[e_+, e_{\bar{1}}]_{\bar{n}} &= [e_+, e_2]_p\epsilon^p{}_{\bar{n}}\,, \\
H_{-\bar{1}\bar{2}} &= -\frac{1}{2}(de_-)_{pq}\epsilon^{pq}\,, \\
[e_-, e_1]_{\bar{n}} &= -[e_-, e_2]_p\epsilon^p{}_{\bar{n}}\,, \\
H_{-+\bar{2}} + H_{1\bar{1}\bar{2}} &= \frac{1}{2}(de_1)_{pq}\epsilon^{pq} + \frac{1}{2}(de_{\bar{1}})_{pq}\epsilon^{pq}\,, \\
[e_-, e_+]_{\bar{n}} + [e_1, e_{\bar{1}}]_{\bar{n}} &= [e_1, e_2]_p\,\epsilon^p{}_{\bar{n}} + [e_{\bar{1}}, e_2]_p\,\epsilon^p{}_{\bar{n}}\,, \\
\partial_{\bar{1}}\Phi - \frac{1}{4}(de_2)_{pq}\,\epsilon^{pq} + \frac{1}{4}(de_{\bar{2}})_{\bar{p}\bar{q}}\,\epsilon^{\bar{p}\bar{q}} &= 0\,, \\
\partial_{\bar{2}}\Phi - \frac{1}{2}(de_{\bar{2}})_p{}^p - \frac{1}{4}(de_{\bar{1}})_{pq}\epsilon^{pq} + \frac{1}{4}(de_1)_{pq}\epsilon^{pq} &= 0\,, \\
\partial_{\bar{n}}\Phi + \frac{1}{2}[e_{\bar{1}}, e_2]_p\epsilon^p{}_{\bar{n}} - \frac{1}{2}[e_1, e_2]_p\epsilon^p{}_{\bar{n}} + \frac{1}{2}[e_2, e_{\bar{2}}]_{\bar{n}} - \frac{1}{2}(\theta_{\omega_I})_{\bar{n}} &= 0\,.
\end{aligned} \tag{12.14}$$

There is no apparent restriction on the rotations $\tilde{de}^a$ unless $\mathfrak{h}$ is abelian in which case $\tilde{de}^+, \tilde{de}^-, \tilde{de}^{\underline{1}} \in \mathfrak{su}(2)$. The dilaton is invariant under e_+, e_- and $e_{\underline{1}}$ $\hat{\nabla}$-parallel vectors.

12.4 N=3

12.4.1 Killing spinors

There are two cases to investigate depending on the choice of the first two Killing spinors. These lead to different results, so they will be examined separately.

Suppose that $\epsilon_1 = 1 + e_{1234}$ and $\epsilon_2 = i(1 - e_{1234})$ and $\mathcal{K}$ is spanned by these two spinors. After some computation one can show that $\mathrm{Stab}(\mathcal{K}) = (\mathrm{Spin}(1,1) \times \mathrm{Spin}(2)_L \times$

$\mathrm{Spin}(2)_R \times \mathrm{Sp}(1)) \ltimes \mathbb{R}^4$. To see how this acts, first write $\mathcal{P}/\mathcal{K} = \mathbb{R}^2 \oplus \mathbb{H}$. Then the action of the subgroup $\mathrm{Spin}(2)_L \times \mathrm{Spin}(2)_R \times \mathrm{Sp}(1)$ is

$$(\underline{x}, y) \longrightarrow (L\underline{x}, ayR^{-1})\,, \quad L \in \mathrm{Spin}(2)_L\,, \quad a \in \mathrm{Sp}(1)\,, \quad R \in \mathrm{Spin}(2)_R\,. \tag{12.15}$$

In particular in the basis (12.1), $\mathfrak{sp}(1) = \mathbb{R} < \frac{i}{2}(\Gamma^{1\bar{1}} - \Gamma^{2\bar{2}}), \Gamma^{1\bar{2}}, \Gamma^{\bar{1}2} >$, $\mathfrak{spin}(2)_L = \mathbb{R} < \frac{i}{2}(\Gamma^{1\bar{1}} + \Gamma^{2\bar{2}}) >$, $\mathfrak{spin}(2)_R = \mathbb{R} < \frac{i}{2}(\Gamma^{3\bar{3}} + \Gamma^{4\bar{4}}) >$, $\mathrm{Spin}(1,1)$ acts with boosts in the Γ_{-+} direction and $\mathbb{R}^4 = \mathbb{R} < \Gamma^{-1}, \Gamma^{-\bar{1}}, \Gamma^{-2}, \Gamma^{-\bar{2}} >$. In addition, $\mathbb{R}^2 = \mathbb{R} < e_{12} - e_{34}, i(e_{12} + e_{34}) >$ and $\mathbb{H}$ spans the rest of the directions. Observe that both $\mathrm{Spin}(2)_L$ and $\mathrm{Spin}(2)_R$ act on $\mathcal{K}$. There are two type of orbits of $\mathrm{Stab}(\mathcal{K})$ in $\mathcal{P}/\mathcal{K}$, one is co-dimension zero in $\mathbb{R}^2$ and the other is codimension zero in $\mathcal{P}/\mathcal{K}$. To see this, observe that the orbit of $\mathrm{Sp}(1)$ in $\mathbb{H}$ is an S^3 sphere and that $\mathrm{Spin}(2)_L \times \mathrm{Spin}(2)_R \times \mathrm{Sp}(1)$ has three types of orbits in $\mathcal{P}/\mathcal{K}$. However, the representatives of two of the orbits are related by an $\mathbb{R}^4$ transformation as in the $N = 2$ case. Choosing representatives for the two orbits of $\mathrm{Stab}(\mathcal{K})$ in $\mathcal{P}/\mathcal{K}$ the third Killing spinor can be chosen either as

$$\epsilon_3 = i(e_{12} + e_{34}) \tag{12.16}$$

or as

$$\epsilon_3 = e_{15} + e_{2345}\,. \tag{12.17}$$

Consequently, the dilatino Killing spinor equation becomes either

$$\mathcal{A}1 = \mathcal{A}(e_{12} + e_{34}) = 0 \tag{12.18}$$

or

$$\mathcal{A}1 = \mathcal{A}(e_{15} + e_{2345}) = 0 \tag{12.19}$$

respectively.

Next suppose that $\epsilon_1 = 1 + e_{1234}$ $\epsilon_2 = e_{15} + e_{2345}$ and that $\mathcal{K}$ is spanned by these two spinors. It turns out that $\mathrm{Stab}(\mathcal{K}) = SL(2,\mathbb{R}) \times SO(3)$, and $SL(2,\mathbb{R}) = \mathrm{Spin}(2,1)$ acts on $\mathcal{K}$ with the two-dimensional representation. To see how this group acts on $\mathcal{P}/\mathcal{K}$ write $\mathcal{P}/\mathcal{K} = \mathbb{R}^3 \oplus \mathbb{R}^3$. Then we have

$$(\mathbf{x}, \mathbf{y}) \longrightarrow (O\mathbf{x}, O\mathbf{y})L^{-1}\,, \quad O \in SO(3)\,, \quad L \in SL(2,\mathbb{R})\,. \tag{12.20}$$

In the basis (12.1), we have that[14] $\mathfrak{sl}(2,\mathbb{R}) = \mathbb{R} < \Gamma^{-+}, \Gamma^{+\underline{1}}, \Gamma^{-\underline{1}} >$, $\mathfrak{so}(3) = \mathbb{R} < \frac{i}{2}(\Gamma^{3\bar{3}} + \Gamma^{4\bar{4}} - 2\Gamma^{2\bar{2}}), \Gamma^{1\bar{2}} - \Gamma^{\bar{1}\bar{2}} - \Gamma^{34}, \Gamma^{\bar{1}2} - \Gamma^{12} - \Gamma^{\bar{3}\bar{4}} >$, and one of the $\mathbb{R}^3$ subspaces is spanned by $\mathbb{R}^3 = \mathbb{R} < i(e_5 - e_{12345}), e_{125} - e_{345}, i(e_{125} + e_{345}) >$ and the other by the rest elements of the basis.

To find the orbits of $\mathrm{Stab}(\mathcal{K})$ consider the invariant

$$I = \mathbf{x}^2\mathbf{y}^2 - (\mathbf{x} \cdot \mathbf{y})^2\,, \tag{12.21}$$

[14] $\Gamma^{\underline{1}}$ denotes the gamma matrix along the real direction 1 to distinguish it from that the complex direction 1 used for the generators of SO(3).

where $\mathbf{x}\cdot\mathbf{y}$ is the Euclidean inner product of $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x}^2=\mathbf{x}\cdot\mathbf{x}$, and similarly for $\mathbf{y}$. If $I\neq 0$, there is a co-dimension one orbit $(\mathrm{SL}(2,\mathbb{R})\times \mathrm{SO}(3))/\mathrm{SO}(2)$ represented by two non-colinear non-vanishing elements $\mathbf{x}$ and $\mathbf{y}$. If $I=0$, then either $\mathbf{y}=0$ or $\mathbf{x}=0$ or $\mathbf{x}$ is colinear to $\mathbf{y}$. In the first two cases, the orbits are codimension zero in the first subspace $\mathbb{R}^3$ or the second subspace $\mathbb{R}^3$, respectively. The latter case is not independent because there is always an $\mathrm{SL}(2,\mathbb{R})$ transformation to transform $\pm(\mathbf{x},\mathbf{x})$ to an element in one of the two $\mathbb{R}^3$ subspaces of $\mathcal{P}/\mathcal{K}$. Therefore there are three types of orbits to consider and the representatives can be chosen as

$$\epsilon_3=i(1-e_{1234})\,, \tag{12.22}$$

or

$$\epsilon_3=i(e_{15}-e_{2345})\,, \tag{12.23}$$

or

$$\epsilon_3=i(1-e_{1234})+(e_{25}-e_{1345})\ . \tag{12.24}$$

In the latter case we have used the freedom to choose the overall scale of the Killing spinor to be one.

To give the independent Killing spinor equations, observe that $\epsilon_1=1+e_{1234}$, $\epsilon_2=e_{15}+e_{2345}$, $\epsilon_3=i(e_{15}-e_{2345})$ and $\epsilon_1=1+e_{1234}$, $\epsilon_2=i(1-e_{1234})$, $\epsilon_3=e_{15}+e_{2345}$ are related by a $\mathrm{Spin}(9,1)$ transformation. Consequently, two of the above three case are related to the two cases described before. Thus the only additional independent dilatino Killing spinor equation is

$$\mathcal{A}(1+e_{1234})=\mathcal{A}(e_{15}+e_{2345})=\mathcal{A}[i(1-e_{1234})+e_{25}-e_{1345}]=0\ . \tag{12.25}$$

12.4.2 $\mathcal{A}1=\mathcal{A}(e_{12}+e_{34})=0$

The solution can be found in [28] and it can be re-expressed in SU(2) representations as

$$\begin{aligned}
\partial_+\Phi&=0\,,\\
H_{+12}=H_{+1\bar1}+H_{+2\bar2}&=0\,,\\
[e_+,e_1]_n=[e_+,e_2]_n=[e_+,e_2]_{\bar n}-[e_+,e_{\bar1}]_p\epsilon^p{}_{\bar n}&=0\,,\\
[e_{\bar1},e_{\bar2}]_n+[e_1,e_{\bar1}]_{\bar p}\epsilon^{\bar p}{}_n+[e_2,e_{\bar2}]_{\bar p}\epsilon^{\bar p}{}_n&=0\,,\\
[e_{\bar1},e_{\bar2}]_{\bar n}&=0\,,\\
(de_+)_{pq}=(de_+)_n{}^n=0\,,\qquad (de_{\bar1})_{\bar n\bar m}=(de_{\bar2})_{\bar n\bar m}&=0\,,\\
\frac{1}{2}(de_2)_{\bar n\bar m}\epsilon^{\bar n\bar m}+(de_{\bar1})_m{}^m=0\,,\quad -\frac{1}{2}(de_1)_{\bar n\bar m}\epsilon^{\bar n\bar m}+(de_{\bar2})_m{}^m&=0\,,\\
\partial_{\bar1}\Phi+\frac{1}{4}(de_2)_{\bar n\bar m}\epsilon^{\bar n\bar m}-\frac{1}{2}H_{2\bar2\bar1}+\frac{1}{2}H_{+-\bar1}&=0\,,\\
\partial_{\bar2}\Phi-\frac{1}{4}(de_1)_{\bar n\bar m}\epsilon^{\bar n\bar m}-\frac{1}{2}H_{1\bar1\bar2}+\frac{1}{2}H_{+-\bar2}&=0\,,\\
\partial_{\bar n}\Phi+\frac{1}{2}[e_{\bar1},e_{\bar2}]_p\epsilon^p{}_{\bar n}-\frac{1}{2}[e_+,e_-]_{\bar n}-\frac{1}{2}(\theta_\omega)_{\bar n}&=0\ .
\end{aligned} \tag{12.26}$$

The dilaton is invariant only under the action of e_+, and $\tilde{d}e^-\in\mathfrak{su}(2)$.

12.4.3 $\mathcal{A}1 = \mathcal{A}(e_{15} + e_{2345}) = 0$

The solution can be easily found by combining (10.12) with (3.12). Expressing the conditions in SU(2) representations, one finds

$$\begin{aligned}
\partial_+\Phi = \partial_-\Phi = \partial_1\Phi &= 0\,, \quad H_{+1\bar{1}} + H_{+2\bar{2}} + (de_+)_n{}^n = 0\,,\\
H_{-1\bar{1}} - H_{-2\bar{2}} - (de_-)_n{}^n &= 0\,, \quad H_{-+\bar{1}} + H_{\bar{1}2\bar{2}} + (de_{\bar{1}})_n{}^n = 0\,,\\
H_{-1\bar{2}} &= -\frac{1}{2}(de_-)_{pq}\epsilon^{pq}\,,\\
[e_-, e_1]_{\bar{n}} &= -[e_-, e_2]_p\epsilon^p{}_{\bar{n}}\,,\\
H_{+12} &= 0\,,\\
[e_+, e_1]_n = [e_+, e_2]_n = [e_1, e_2]_n &= 0\,,\\
(de_+)_{pq} = (de_2)_{pq} = (de_1)_{pq} &= 0\,,\\
H_{-+\bar{2}} &= \frac{1}{2}(de_{\bar{1}})_{pq}\epsilon^{pq} - H_{\bar{2}1\bar{1}}\,,\\
[e_-, e_+]_{\bar{n}} &= [e_{\bar{1}}, e_2]_p\epsilon^p{}_{\bar{n}} - [e_1, e_{\bar{1}}]_{\bar{n}}\,,\\
\partial_{\bar{2}}\Phi - \frac{1}{2}(de_{\bar{2}})_p{}^p - \frac{1}{4}(de_{\bar{1}})_{pq}\epsilon^{pq} &= 0\,,\\
\partial_{\bar{n}}\Phi + \frac{1}{2}[e_2, e_{\bar{2}}]_n + \frac{1}{2}[e_{\bar{1}}, e_2]_p\epsilon^p{}_{\bar{n}} - \frac{1}{2}(\theta_\omega)_{\bar{n}} &= 0\,,
\end{aligned} \tag{12.27}$$

The anti-self dual part of $\tilde{de}^a$, for $a = -, +, 1$ i.e. the (2,0)+(0,2) and hermitian trace, is entirely expressed in terms of the structure constants of $\mathfrak{h}$. Therefore if $\mathfrak{h}$ is abelian the $\tilde{de}^-, \tilde{de}^+, \tilde{de}^1$ take values in $\mathfrak{su}(2)$. The dilaton is invariant under four of the six parallel vectors.

12.4.4 $\mathcal{A}(1 + e_{1234}) = \mathcal{A}(e_{15} + e_{2345}) = \mathcal{A}[i(1 - e_{1234}) + e_{25} - e_{1345}] = 0$

The solution of the first two conditions can be found in (3.12). The third condition is new. The solution of the dilatino Killing spinor equations expressed in SU(2) representations is

$$\begin{aligned}
&\partial_-\Phi = \partial_+\Phi = \partial_1\Phi = \partial_2\Phi = 0\,,\\
&([e_1, e_2]_m + [e_{\bar{1}}, e_2]_m)\epsilon^m{}_{\bar{n}} = [e_-, e_+]_{\bar{n}} + [e_1, e_{\bar{1}}]_{\bar{n}}\,,\\
&([e_1, e_{\bar{2}}]_p - [e_{\bar{1}}, e_2]_p)\epsilon^p{}_{\bar{n}} = -[e_1, e_{\bar{1}}]_{\bar{n}} + [e_2, e_{\bar{2}}]_{\bar{n}} + 2i[e_+, e_{\bar{2}}]_{\bar{n}}\,,\\
&i([e_-, e_2]_{\bar{m}} - [e_-, e_1]_p\epsilon^p{}_{\bar{m}}) = 2[e_1, e_2]_p\epsilon^p{}_{\bar{m}}\,,\\
&[e_+, e_{\bar{1}}]_{\bar{m}} - [e_+, e_2]_n\epsilon^n{}_{\bar{m}} = [e_-, e_1]_{\bar{m}} + [e_-, e_2]_n\epsilon^n{}_{\bar{m}} = 0\,,\\
&\epsilon^{mn}(de_+)_{mn} = 2H_{+\bar{1}\bar{2}}\,, \quad (de_+)_n{}^n = -H_{+1\bar{1}} - H_{+2\bar{2}}\,,\\
&\epsilon^{mn}(de_-)_{mn} = -2H_{-1\bar{2}}\,, \quad (de_-)_n{}^n = H_{-1\bar{1}} - H_{-2\bar{2}}\,,\\
&(de_{\bar{1}})_n{}^n = -H_{-+\bar{1}} - H_{\bar{1}2\bar{2}} + iH_{-1\bar{2}} - iH_{-\bar{1}2}\,,\\
&(de_1)_{mn}\epsilon^{mn} = -iH_{-1\bar{1}} + iH_{-2\bar{2}}\,,\\
&(de_{\bar{1}})_{mn}\epsilon^{mn} = 2H_{-+\bar{2}} + 2H_{\bar{2}1\bar{1}} + iH_{-1\bar{1}} - iH_{-2\bar{2}}\,,\\
&\epsilon^{mn}(de_2)_{mn} = -i(H_{-1\bar{2}} - H_{-\bar{1}2})\,,\\
&(de_{\bar{2}})_n{}^n = -H_{-+\bar{2}} - H_{\bar{2}1\bar{1}} - iH_{-1\bar{1}} + iH_{-2\bar{2}}\,,\\
&\epsilon^{mn}(de_{\bar{2}})_{mn} = -2H_{-+\bar{1}} - 2H_{\bar{1}2\bar{2}} + i(H_{-1\bar{2}} - H_{-\bar{1}2}) + 4iH_{+12}\,, \qquad (12.28)\\
&0 = \partial_{\bar{n}}\Phi + \frac{1}{2}[e_2, e_{\bar{2}}]_{\bar{n}} + \frac{1}{2}[e_1, e_{\bar{1}}]_{\bar{n}} + \frac{1}{2}[e_-, e_+]_{\bar{n}} - \frac{1}{2}(\theta_{\omega_I})_{\bar{n}} - [e_1, e_2]_m\epsilon^m{}_{\bar{n}}.
\end{aligned}$$

The anti-self dual part of $\tilde{de}^a$ is entirely expressed in terms of the structure constants of $\mathfrak{h}$. The dilaton is invariant under all parallel vector fields. Observe that if $\mathfrak{h}$ is abelian, then the above conditions are the same as those that one can derive from the dilatino Killing spinor equation of $N=8$ backgrounds [28], see also (12.59). So the supersymmetry enhances to[15] $N=8$.

12.5 N=4

12.5.1 Killing spinors

To begin, suppose that $\epsilon_1 = 1 + e_{1234}$, $\epsilon_2 = i(1 - e_{1234})$, $\epsilon_3 = i(e_{12} + e_{34})$ and that $\mathcal{K}$ is spanned by these three spinors. Then, $\mathrm{Stab}(\mathcal{K}) = (\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R \times \mathrm{Spin}(1,1)) \ltimes \mathbb{R}^4$. Writing $\mathcal{P}/\mathcal{K} = \mathbb{R} \oplus \mathbb{H}$, the subgroup $\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R$ acts only on $\mathbb{H}$ as $y \to a y \bar{b}$, where $a \in \mathrm{Sp}(1)_L$ and $b \in \mathrm{Sp}(1)_R$. In the basis (12.1), $\mathfrak{sp}(1)_L = \mathbb{R} < \frac{i}{2}(\Gamma^{1\bar{1}} - \Gamma^{2\bar{2}}), \Gamma^{1\bar{2}}, \Gamma^{\bar{1}2} >$, $\mathfrak{sp}(1)_R = \mathbb{R} < \frac{i}{2}(\Gamma^{1\bar{1}} + \Gamma^{2\bar{2}} - \Gamma^{3\bar{3}} - \Gamma^{4\bar{4}}), \Gamma^{\bar{1}\bar{2}} - \Gamma^{34}, \Gamma^{12} - \Gamma^{\bar{3}\bar{4}} >$, $\mathrm{Spin}(1,1)$ is generated by boosts along Γ^{+-} and the Lie algebra of $\mathbb{R}^4$ is generated by $\mathbb{R}^4 = \mathbb{R} < \Gamma^{-1}, \Gamma^{-\bar{1}}, \Gamma^{-2}, \Gamma^{-\bar{2}} >$. In addition if the subspace $\mathbb{R}$ of $\mathcal{P}/\mathcal{K}$ is chosen along the $i(e_{12} + e_{34})$ direction, $\mathbb{H}$ spans the rest of the directions. Using a similar argument as in previous cases, it is easy to see that $\mathrm{Stab}(\mathcal{K})$ has two types of orbits in $\mathcal{P}/\mathcal{K}$ one has codimension zero in $\mathbb{R}$ and the other has codimension zero in $\mathcal{P}/\mathcal{K}$. So the forth Killing spinor can be chosen either as

$$\epsilon_4 = e_{12} - e_{34}\,, \tag{12.29}$$

or as

$$\epsilon_4 = e_{15} + e_{2345}\ . \tag{12.30}$$

So the dilatino Killing spinor equation is either

$$\mathcal{A}1 = \mathcal{A}e_{12} = 0 \tag{12.31}$$

or

$$\mathcal{A}1 = \mathcal{A}(e_{12} + e_{34}) = \mathcal{A}(e_{15} + e_{2345}) = 0 \tag{12.32}$$

Next suppose that $\epsilon_1 = 1 + e_{1234}$, $\epsilon_2 = i(1 - e_{1234})$, $\epsilon_3 = e_{15} + e_{2345}$ and that $\mathcal{K}$ is spanned by these three spinors. It turns out that $\mathrm{Stab}(\mathcal{K}) = (\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{Spin}(1,1)) \ltimes \mathbb{R}^2$. In the basis (12.1), $\mathfrak{u}(1) \oplus \mathfrak{u}(1) = \mathbb{R} < \frac{i}{4}(\Gamma^{1\bar{1}} - \Gamma^{2\bar{2}} + \Gamma^{3\bar{3}} + \Gamma^{4\bar{4}}), \frac{i}{2}(\Gamma^{1\bar{1}} + \Gamma^{2\bar{2}}) >$, $\mathrm{Spin}(1,1)$ is generated by the boosts Γ^{+-} and $\mathbb{R}^2$ is generated by $\Gamma^{-\underline{1}}, \Gamma^{-\underline{6}}$. Writing $\mathcal{P}/\mathcal{K} = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$, where the former $\mathbb{R}^2$ is spanned by $e_{12} - e_{34}, i(e_{12} + e_{34})$, and the latter is spanned by $e_{25} - e_{1345}, i(e_{25} + e_{1345})$ and $\mathbb{R}$ by the remaining direction. Each $\mathrm{U}(1)$ acts on a $\mathbb{R}^2$ with the two-dimensional representation. There are several types of orbits which can be represented by

$$\epsilon_4 = i(e_{12} + e_{34})\,, \tag{12.33}$$

$$\epsilon_4 = i(e_{15} - e_{2345})\,, \tag{12.34}$$

$$\epsilon_4 = i(e_{12} + e_{34}) + i(e_{15} - e_{2345}) \tag{12.35}$$

[15] Note that this case, together with the subsequent cases where requiring $\mathfrak{h}$ to be abelian implies $N=8$, are exactly those which have $\mathrm{Stab}_\Sigma = 1$.

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and

$$\epsilon_4 = \cos\varphi(e_{25} - e_{1345}) + i\sin\varphi(e_{15} - e_{2345}) \tag{12.36}$$

Only the latter three choices give independent new cases. The dilatino Killing spinor equations are

$$\mathcal{A}1 = \mathcal{A}e_{15} = 0\,, \tag{12.37}$$
$$\mathcal{A}1 = \mathcal{A}(e_{15} + e_{2345}) = \mathcal{A}(e_{12} + e_{34} + (e_{15} - e_{2345})) = 0\,, \tag{12.38}$$
$$\mathcal{A}1 = \mathcal{A}(e_{15} + e_{2345}) = \mathcal{A}(\cos\varphi(e_{25} - e_{1345}) + i\sin\varphi(e_{15} - e_{2345})) = 0\,, \tag{12.39}$$

Suppose that $\epsilon_1 = 1 + e_{1234}$, $\epsilon_2 = e_{15} + e_{2345}$, $\epsilon_3 = i(1 - e_{1234}) + e_{25} - e_{1345}$ and that $\mathcal{K}$ is spanned by these three spinors. One can show that $\mathrm{Stab}(\mathcal{K}) = SO(3)$ and acts on $\mathcal{P}/\mathcal{K}$ with the symmetric traceless product of the vector representation. In the basis (12.1), $\mathfrak{so}(3) = \mathbb{R} < t_1, t_2, [t_1, t_2] >$, where

$$\begin{aligned} t_1 &= \frac{1}{4}\Big[3\Gamma^{1\bar{2}} + 3\Gamma^{\bar{1}2} - \Gamma^{\bar{1}\bar{2}} - \Gamma^{12} - \Gamma^{34} - \Gamma^{\bar{3}\bar{4}}\Big] + \frac{1}{\sqrt{2}}\Gamma^{-\underline{6}} \\ t_2 &= \frac{i}{4}\Big[2\Gamma^{1\bar{1}} + 4\Gamma^{2\bar{2}} - \Gamma^{3\bar{3}} - \Gamma^{4\bar{4}}\Big] - \frac{1}{\sqrt{2}}\Gamma^{+\underline{2}}\;. \end{aligned} \tag{12.40}$$

From this one can easily show that the above generators satisfy the Lie algebra relations of SO(3). To identify $\mathcal{P}/\mathcal{K}$ with the traceless symmetric representation, $S^2_0(\mathbb{R}^3)$, first observe that SU(2) has real representations of dimensions three, four and five, and all the rest are of higher dimension. In the first two cases $\mathcal{P}/\mathcal{K}$ would have been the sum of an irreducible and trivial representations. This means that $\mathcal{P}/\mathcal{K}$ would have a one-dimensional invariant subspace under the SO(3) action. However, one can easily show that such a subspace does not exit. Thus the only other option available is to identify $\mathcal{P}/\mathcal{K} = S^2_0(\mathbb{R}^3)$. A direct computation in appendix C has confirmed this. There are two types of orbits of SO(3) in $S^2_0(\mathbb{R}^3)$. One is a generic orbit of co-dimension two isomorphic to SO(3) and the other is a special S^2 orbit. This can be easily seen by observing that any 3×3 symmetric traceless matrix can be diagonalized and has two eigenvalues. If the two eigenvalues are distinct, then the symmetric matrix represent the generic orbit. If either one of the eigenvalues vanishes or their sum vanishes, then the symmetric matrix represents the special S^2 orbit. A representative of the special orbit can be identified with a spinor that is invariant under one of the generators of SO(3). Therefore, we can choose as a fourth Killing spinor either

$$\epsilon_4 = i\cos\varphi(e_{15} - e_{2345}) + i\sin\varphi(e_{12} + e_{34}) \tag{12.41}$$

where φ is a constant angle, or

$$\epsilon_4 = i(e_{12} + e_{34})\;. \tag{12.42}$$

Thus the dilatino Killing spinor equation is either

$$\begin{aligned} \mathcal{A}(1 + e_{1234}) &= \mathcal{A}(e_{15} + e_{2345}) = \mathcal{A}(i(1 - e_{1234}) + e_{25} - e_{1345}) \\ &= \mathcal{A}[\cos\varphi(e_{15} - e_{2345}) + \sin\varphi(e_{12} + e_{34})] = 0\,, \end{aligned} \tag{12.43}$$

or

$$\mathcal{A}(1+e_{1234})=\mathcal{A}(e_{15}+e_{2345})=\mathcal{A}(i(1-e_{1234})+e_{25}-e_{1345})=\mathcal{A}(e_{12}+e_{34})=0\,. \quad (12.44)$$

The latter case is a special case of the former for $\sin\varphi=1$.

12.5.2 $\mathcal{A}1=\mathcal{A}e_{12}=0$

The solution of the dilatino Killing spinor equation expressed in SU(2) representations is

$$\begin{aligned}
\partial_+\Phi=0\,,\quad H_{+\alpha\beta}=H_{+\alpha}{}^{\alpha}&=0\,,\quad [e_+,e_\alpha]_i=0\,,\quad [e_\alpha,e_\beta]_i=[e_\alpha,e^\alpha]_i=0\,,\\
(de_+)_{np}=(de_+)_n{}^n&=0\,,\\
(de_\alpha)_{np}=(de_\alpha)_{\bar n\bar p}=(de_\alpha)_n{}^n&=0\,,\\
\partial_{\bar\alpha}\Phi-\frac{1}{2}H_{\bar\alpha\beta}{}^{\beta}-\frac{1}{2}H_{-+\bar\alpha}&=0\,,\\
\partial_{\bar n}\Phi-\frac{1}{2}\theta_{\bar n}+\frac{1}{2}[e_-,e_+]_{\bar n}&=0\,,\quad \alpha=1,2\,,\quad n,p=3,4\,,\quad i=3,4,\bar 3,\bar 4. \quad (12.45)
\end{aligned}$$

If $\mathfrak{h}$ is not abelian, the dilaton is invariant under e_+, $\tilde{d}e^-,\tilde{d}e^\alpha\in\mathfrak{su}(2)$ but $\tilde{d}e^-$ is not restricted. For abelian $\mathfrak{h}$, the dilaton is invariant under e_+,e_1,e_2.

12.5.3 $\mathcal{A}1=\mathcal{A}(e_{12}+e_{34})=\mathcal{A}(e_{15}+e_{2345})=0$

The solution of the dilatino Killing spinor equation is

$$\begin{aligned}
\partial_-\Phi=\partial_+\Phi=\partial_1\Phi=(\partial_2-\partial_{\bar 2})\Phi&=0\,,\\
H_{+12}=H_{+1\bar 1}+H_{+2\bar 2}&=0\,,\\
[e_1,e_2]_n=[e_+,e_1]_n=[e_+,e_2]_n=[e_1,e_{\bar 1}]_{\bar n}+[e_-,e_+]_{\bar n}-[e_{\bar 1},e_2]_p\epsilon^p{}_{\bar n}&=0\,,\\
[e_1,e_{\bar 1}]_{\bar n}+[e_2,e_{\bar 2}]_{\bar n}-[e_{\bar 1},e_{\bar 2}]_p\epsilon^p{}_{\bar n}=[e_+,e_2]_{\bar n}-[e_+,e_{\bar 1}]_p\epsilon^p{}_{\bar n}=[e_-,e_1]_{\bar n}+[e_-,e_2]_p\epsilon^p{}_{\bar n}&=0\,,\\
(de_+)_n{}^n=(de_+)_{pq}&=0\,,\\
H_{-1\bar 1}-H_{-2\bar 2}-(de_-)_n{}^n&=0\,,\\
H_{-1\bar 2}+\frac{1}{2}(de_-)_{pq}\epsilon^{pq}&=0\,,\\
H_{21\bar 1}+H_{+-2}+\frac{1}{2}\epsilon^{\bar m\bar n}(de_1)_{\bar m\bar n}=(de_1)_{mn}=(de_{\bar 1})_n{}^n+H_{-+\bar 1}+H_{\bar 12\bar 2}&=0\,,\\
H_{+-\bar 1}-H_{\bar 12\bar 2}+\frac{1}{2}\epsilon^{\bar m\bar n}(de_2)_{\bar m\bar n}=H_{21\bar 1}+H_{+-2}+(de_{\bar 2})_p{}^p=(de_2)_{mn}&=0\,,\\
2\partial_{\bar 2}\Phi+H_{+-2}+H_{+-\bar 2}+H_{21\bar 1}-H_{\bar 21\bar 1}&=0\,,\\
2\partial_{\bar n}\Phi+[e_1,e_{\bar 1}]_{\bar n}+[e_2,e_{\bar 2}]_{\bar n}-[e_+,e_-]_{\bar n}-(\theta_\omega)_{\bar n}&=0\,. \quad (12.46)
\end{aligned}$$

The dilaton is invariant under five parallel vector field. Moreover $\tilde{d}e^-\in\mathfrak{su}(2)$ and the anti-self dual part of $\tilde{d}e^+$, $\tilde{d}e^1$ and $\tilde{d}e^2$ are determined in terms of the structure constants of $\mathfrak{h}$. So if $\mathfrak{h}$ is abelian, all rotations are in $\mathfrak{su}(2)$. In addition Φ is invariant under all parallel vectors fields. As a consequence, there is supersymmetry enhancement to $N=8$.

12.5.4 $\mathcal{A}1 = \mathcal{A}e_{15} = 0$

The solution in SU(2) representations is

$$\begin{aligned}
\partial_+\Phi = \partial_-\Phi = \partial_1\Phi = 0\,, \quad H_{-1\bar{2}} = H_{+12} = H_{-+\bar{2}} + H_{\bar{2}1\bar{1}} &= 0\,, \\
(de_+)_{np} = 0\,, \qquad (de_+)_n{}^n + H_{+2\bar{2}} + H_{+1\bar{1}} &= 0\,, \\
(de_-)_{np} = 0\,, \qquad H_{-1\bar{1}} - H_{-2\bar{2}} - (de_-)_n{}^n &= 0\,, \\
(de_{\bar{1}})_{np} = (de_1)_{np} = 0\,, \qquad (de_{\bar{1}})_n{}^n + H_{\bar{1}2\bar{2}} + H_{-+\bar{1}} &= 0\,, \\
(de_2)_{np} = 0\,, \qquad [e_-, e_+]_{\bar{n}} + [e_1, e_{\bar{1}}]_{\bar{n}} &= 0\,, \\
[e_+, e_2]_n = [e_+, e_1]_n = [e_-, e_2]_n = [e_-, e_{\bar{1}}]_n = [e_{\bar{1}}, e_2]_n = [e_1, e_2]_n &= 0 \\
\partial_{\bar{2}}\Phi - \frac{1}{2}(de_{\bar{2}})_n{}^n = 0\,, \qquad \partial_{\bar{n}}\Phi - \frac{1}{2}\theta_{\bar{n}} + \frac{1}{2}[e_2, e_{\bar{2}}]_{\bar{n}} &= 0\,. \qquad (12.47)
\end{aligned}$$

The dilaton is invariant under e_+, e_- and e_1. Moreover $\tilde{de}^a \in \mathfrak{u}(2)$, $a = -, +, 1, 2$. For the first three rotations, the hermitian trace depends on the structure constants of $\mathfrak{h}$. Consequently even if $\mathfrak{h}$ is abelian, the hermitian trace of $\tilde{de}^2$ does not vanish. There is no supersymmetry enhancement to $N = 8$.

12.5.5 $\mathcal{A}1 = \mathcal{A}(e_{15} + e_{2345}) = \mathcal{A}(e_{12} + e_{34} + (e_{15} - e_{2345})) = 0$

The solution for the first three Killing spinor equations is given in (12.27). The forth Killing spinor equation gives the additional constraints

$$\begin{aligned}
(de_+)_m{}^m &= -2H_{\bar{2}1\bar{1}} - 2H_{-+\bar{2}} = 2(de_{\bar{2}})_m{}^m\,, \\
\epsilon^{\bar{m}\bar{n}}(de_2)_{\bar{m}\bar{n}} &= 4H_{-\bar{1}2} + 2H_{-+\bar{1}} + 2H_{\bar{1}2\bar{2}}\,, \\
[e_{\bar{1}}, e_{\bar{2}}]_p\epsilon^p{}_{\bar{n}} &= [e_1, e_{\bar{1}}]_{\bar{n}} + [e_2, e_{\bar{2}}]_{\bar{n}} + 2[e_-, e_{\bar{2}}]_{\bar{n}}\,, \\
[e_+, e_2]_{\bar{m}} + \epsilon_{\bar{m}}{}^p[e_+, e_{\bar{1}}]_p &= 2[e_-, e_+]_{\bar{n}} + 2[e_1, e_{\bar{1}}]_{\bar{n}}\,. \qquad (12.48)
\end{aligned}$$

Note that these conditions together with (12.27) imply that $\partial_2\Phi = 0$. Therefore the dilaton is invariant under all parallel vector fields. The anti-self dual part of $\tilde{de}^a$, for $a = -, +, 1, 2$ is entirely expressed in terms of the structure constants of $\mathfrak{h}$. Hence, if $\mathfrak{h}$ is abelian then $\tilde{de}^a$ takes values in $\mathfrak{su}(2)$ and supersymmetry enhances to $N = 8$.

12.5.6 $\mathcal{A}1 = \mathcal{A}(e_{15} + e_{2345}) = \mathcal{A}(\cos\varphi(e_{25} - e_{1345}) + i\sin\varphi(e_{15} - e_{2345})) = 0$

The solution for the first three Killing spinor equations is given in (12.27) while the forth implies the additional constraints

$$\begin{aligned}
H_{-1\bar{2}} &= H_{-\bar{1}2}\,, \\
\cos\varphi(de_-)_n{}^n - i\sin\varphi(H_{-1\bar{2}} + H_{-\bar{1}2}) &= 0\,, \\
(de_{\bar{2}})_m{}^m + H_{\bar{2}-+} + H_{\bar{2}1\bar{1}} &= 0\,, \\
\cos\varphi\left(\frac{1}{2}(de_{\bar{2}})_{mn}\epsilon^{mn} + H_{\bar{1}-+} + H_{\bar{1}2\bar{2}}\right) - 2i\sin\varphi(H_{-+\bar{2}} + H_{\bar{2}1\bar{1}}) &= 0\,, \\
\cos\varphi([e_2, e_{\bar{2}}]_{\bar{m}} + [e_-, e_+]_{\bar{m}} - [e_1, e_{\bar{2}}]_p\epsilon^p{}_{\bar{m}}) - 2i\sin\varphi([e_-, e_+]_n - [e_1, e_{\bar{1}}]_n)\epsilon^n{}_{\bar{m}} &= 0\,, \\
\cos\varphi(-[e_-, e_2]_{\bar{m}} + [e_-, e_1]_p\epsilon^p{}_{\bar{m}}) - 2i\sin\varphi[e_-, e_1]_{\bar{m}} &= 0\,, (12.49)
\end{aligned}$$

where we have assumed that $\cos\varphi$ and $\sin\varphi$ do not vanish. In the special case in which $\sin\varphi = 0$, the fourth Killing spinor equation gives the additional constraints

$$\begin{aligned} H_{-1\bar{2}} = H_{-\bar{1}2}\,, \quad (de_-)_n{}^n &= 0\,, \\ (de_{\bar{2}})_m{}^m + H_{\bar{2}-+} + H_{\bar{2}1\bar{1}} = \tfrac{1}{2}(de_{\bar{2}})_{mn}\epsilon^{mn} + H_{\bar{1}-+} + H_{\bar{1}2\bar{2}} &= 0\,, \\ [e_2, e_{\bar{2}}]_{\bar{m}} + [e_-, e_+]_{\bar{m}} - [e_1, e_{\bar{2}}]_p\epsilon^p{}_{\bar{m}} = -[e_-, e_2]_{\bar{m}} + [e_-, e_1]_p\epsilon^p{}_{\bar{m}} &= 0\,. \end{aligned} \tag{12.50}$$

The additional constraints from the fourth Killing spinor equation imply both in the generic and in the special case that $\partial_2\Phi = 0$. Therefore we conclude that in both cases the dilation is invariant under all parallel vectors e_a. The anti-self dual part of all rotations $\tilde{de}^a$ depends on the structure constants of $\mathfrak{h}$. Therefore if $\mathfrak{h}$ is abelian, then $\tilde{de}^a$ is self-dual, i.e. takes values in $\mathfrak{su}(2)$. In such a case, supersymmetry enhances to $N = 8$.

12.5.7 $\mathcal{A}(1+e_{1234}) = \mathcal{A}(e_{15}+e_{2345}) = \mathcal{A}(i(1-e_{1234})+e_{25}-e_{1345}) = \mathcal{A}[\cos\varphi(e_{15}-e_{2345})+\sin\varphi(e_{12}+e_{34})] = 0$

The solution for the first three Killing spinor equations has been given in (12.28). In addition, the (generic) fourth Killing spinor equation, assuming both $\sin\varphi$ and $\cos\varphi$ do not vanish, gives the additional constraints

$$\begin{aligned} H_{-1\bar{2}} - H_{-\bar{1}2} = H_{+12} + H_{+\bar{1}\bar{2}} &= 0\,, \\ H_{-+2} + H_{-+\bar{2}} + H_{\bar{2}1\bar{1}} - H_{21\bar{1}} + i(H_{-1\bar{1}} - H_{-2\bar{2}}) &= 0\,, \\ \sin\varphi(H_{+1\bar{1}} + H_{+2\bar{2}}) - \cos\varphi(H_{-+\bar{2}} - H_{-+2} + H_{\bar{2}1\bar{1}} + H_{21\bar{1}}) &= 0\,, \\ \cos\varphi(H_{-1\bar{2}} + H_{-\bar{1}2}) + \sin\varphi(-H_{12\bar{2}} + H_{\bar{1}2\bar{2}} + H_{-+1} + H_{-+\bar{1}}) &= 0\,, \\ \sin\varphi([e_{\bar{1}}, e_{\bar{2}}]_m + [e_1, e_2]_m - \epsilon_m{}^{\bar{n}}([e_1, e_{\bar{1}}]_{\bar{n}} + [e_2, e_{\bar{2}}]_{\bar{n}})) - 2\cos\varphi[e_-, e_{\bar{1}}]_m &= 0\,, \\ \sin\varphi([e_+, e_2]_{\bar{m}} + \epsilon_{\bar{m}}{}^p[e_+, e_{\bar{1}}]_p) - 2\cos\varphi[e_{\bar{1}}, e_2]_p\epsilon^p{}_{\bar{m}} &= 0\,. \end{aligned} \tag{12.51}$$

Moreover for the special orbit that corresponds to $\sin\varphi = 1$, we find that the solution to fourth Killing spinor equations is

$$\begin{aligned} H_{+1\bar{1}} + H_{+2\bar{2}} = H_{+12} + H_{+\bar{1}\bar{2}} &= 0\,, \\ H_{-+2} + H_{-+\bar{2}} + H_{\bar{2}1\bar{1}} - H_{21\bar{1}} + i(H_{-1\bar{1}} - H_{-2\bar{2}}) &= 0\,, \\ -H_{12\bar{2}} + H_{\bar{1}2\bar{2}} + H_{-+1} + H_{-+\bar{1}} - i(H_{-1\bar{2}} - H_{-\bar{1}2}) &= 0\,, \\ [e_{\bar{1}}, e_{\bar{2}}]_m + [e_1, e_2]_m - \epsilon_m{}^{\bar{n}}([e_1, e_{\bar{1}}]_{\bar{n}} + [e_2, e_{\bar{2}}]_{\bar{n}}) &= 0\,, \\ [e_+, e_2]_{\bar{m}} + \epsilon_{\bar{m}}{}^p[e_+, e_{\bar{1}}]_p &= 0\,. \end{aligned} \tag{12.52}$$

It is easy to see that the conditions in both cases restrict the commutators of the vectors fields e_a and the structure constants of $\mathfrak{h}$. As in the (12.28), the anti-self dual part of $\tilde{de}^a$ is entirely expressed in terms of the structure constants of $\mathfrak{h}$, and the dilaton is invariant under all parallel vector fields. As a consequence, if $\mathfrak{h}$ is abelian, supersymmetry enhances to $N = 8$.

12.6 N=5

12.6.1 Killing spinors

In this case, it is more convenient to use the gauge symmetry to determine the normals to the Killing spinors. The correspondence $N \leftrightarrow 8 - N$ suggests that the normals can be chosen in a way similar to the Killing spinors for $N = 3$ backgrounds. In turn these can be used to find the Killing spinors of the theory. In particular as for $N = 3$ supersymmetric backgrounds, there are three cases to consider.

12.6.2 $\mathcal{A}1 = \mathcal{A}(e_{15} + e_{2345}) = \mathcal{A}e_{12} = 0$

The solution of the Killing spinor equations is

$$\begin{aligned}
\partial_+\Phi = \partial_-\Phi = \partial_1\Phi = \partial_2\Phi &= 0\,,\\
\partial_{\bar{n}}\Phi - \frac{1}{2}(\theta_{\omega_I})_{\bar{n}} + \tfrac{1}{2}[e_-, e_+]_{\bar{n}} &= 0\,,\\
H_{+12} = H_{+1\bar{1}} + H_{+2\bar{2}} = H_{-+\bar{1}} + H_{\bar{1}2\bar{2}} = H_{-+\bar{2}} + H_{\bar{2}1\bar{1}} &= 0\,,\\
[e_+, e_1]_m = [e_+, e_2]_m = [e_+, e_1]_{\bar{m}} = [e_+, e_2]_{\bar{m}} = [e_-, e_1]_{\bar{n}} + [e_-, e_2]_m\epsilon^m{}_{\bar{n}} &= 0\,,\\
[e_1, e_2]_{\bar{m}} = [e_1, e_{\bar{1}}]_m + [e_2, e_{\bar{2}}]_m = [e_-, e_+]_{\bar{m}} + [e_1, e_{\bar{1}}]_{\bar{m}} - [e_{\bar{1}}, e_2]_p\epsilon^p{}_{\bar{m}} &= 0\,,\\
(de_+)_{mn} = (de_+)_n{}^n &= 0\,,\\
(de_-)_n{}^n - H_{-1\bar{1}} + H_{-2\bar{2}} = \frac{1}{2}\epsilon^{mn}(de_-)_{mn} + H_{-1\bar{2}} &= 0\,,\\
(de_1)_{mn} = (de_{\bar{1}})_{mn} = (de_1)_n{}^n &= 0\,,\\
(de_2)_{mn} = (de_{\bar{2}})_{mn} = (de_2)_n{}^n &= 0\,. \qquad (12.53)
\end{aligned}$$

The dilaton is invariant under all parallel vector field e_a. Moreover $\tilde{d}e^a$, $a = -,+,1$, is self-dual, i.e. takes values in $\mathfrak{su}(2)$, while $\tilde{d}e^2$ takes values in $\mathfrak{u}(2)$. The hermitian trace of the latter depends on the structure constants of $\mathfrak{h}$. Therefore if $\mathfrak{h}$ is abelian, there is supersymmetry enhancement to $N = 8$.

12.6.3 $\mathcal{A}1 = \mathcal{A}(e_{12} + e_{34}) = \mathcal{A}e_{15} = 0$

The solution to the Killing spinor equations is

$$\begin{aligned}
\partial_+\Phi = \partial_-\Phi = \partial_1\Phi = \partial_2\Phi &= 0\,,\\
\partial_{\bar{n}}\Phi - \frac{1}{2}(\theta_{\omega_I})_{\bar{n}} + \frac{1}{2}[e_2, e_{\bar{2}}]_{\bar{n}} &= 0\,,\\
H_{+12} = H_{+1\bar{1}} + H_{+2\bar{2}} = H_{-1\bar{2}} = H_{-+\bar{2}} + H_{\bar{2}1\bar{1}} &= 0\,,\\
[e_+, e_1]_m = [e_+, e_2]_m = [e_+, e_2]_{\bar{m}} + \epsilon_{\bar{m}}{}^n[e_+, e_{\bar{1}}]_n = [e_-, e_1]_{\bar{n}} = [e_-, e_2]_m &= 0\,,\\
[e_1, e_2]_m = [e_{\bar{1}}, e_2]_m = [e_1, e_{\bar{1}}]_{\bar{m}} + [e_2, e_{\bar{2}}]_{\bar{m}} + \epsilon_{\bar{m}}{}^p[e_{\bar{1}}, e_{\bar{2}}]_p = [e_-, e_+]_{\bar{m}} + [e_1, e_{\bar{1}}]_{\bar{m}} &= 0\,,\\
(de_+)_{mn} = (de_+)_n{}^n &= 0\,,\\
(de_-)_n{}^n - H_{-1\bar{1}} + H_{-2\bar{2}} = (de_-)_{mn} &= 0\,,\\
(de_1)_{mn} = (de_{\bar{1}})_{mn} = (de_{\bar{1}})_n{}^n + H_{-+\bar{1}} + H_{\bar{1}2\bar{2}} &= 0\,,\\
(de_2)_{mn} = \frac{1}{2}(de_2)_{\bar{m}\bar{n}}\epsilon^{\bar{m}\bar{n}} - H_{-+\bar{1}} - H_{\bar{1}2\bar{2}} = (de_2)_n{}^n &= 0\,. \qquad (12.54)
\end{aligned}$$

The dilaton is invariant under all parallel vector field e_a, and $\tilde{d}e^-$ is self-dual. Moreover $\tilde{d}e^a$, $a = +,1$ takes values in $\mathfrak{u}(2)$, where the hermitian trace depends on the structure constants of $\mathfrak{h}$. Similarly all the anti-self-dual components $\tilde{d}e^2$ depend on the structure constants of $\mathfrak{h}$. Therefore if $\mathfrak{h}$ is abelian, there is supersymmetry enhancement to $N = 8$.

12.6.4 $\mathcal{A}1 = \mathcal{A}e_{15} = \mathcal{A}(e_{25} - e_{1345} + i(e_{12} + e_{34})) = 0$

The first four Killing spinor equations give the conditions (12.47), while the fifth implies the additional constraints

$$\begin{aligned}
(de_-)_n{}^n - (de_+)_n{}^n = (de_1)_n{}^n - (de_{\bar{1}})_n{}^n = (de_2)_n{}^n &= 0\,,\\
\frac{1}{2}(de_2)_{\bar{m}\bar{n}}\epsilon^{\bar{m}\bar{n}} + (de_1)_n{}^n - i(de_-)_n{}^n &= 0\,,\\
-[e_-, e_{\bar{2}}]_m + [e_-, e_{\bar{1}}]_{\bar{p}}\epsilon^{\bar{p}}{}_m - i([e_{\bar{1}}, e_{\bar{2}}]_m - \epsilon_m{}^{\bar{n}}([e_1, e_{\bar{1}}]_{\bar{n}} + [e_2, e_{\bar{2}}]_{\bar{n}}) &= 0\,,\\
-[e_+, e_1]_{\bar{m}} + \epsilon_{\bar{m}}{}^p[e_+, e_{\bar{2}}]_p + i(-[e_1, e_{\bar{1}}]_{\bar{n}} + [e_2, e_{\bar{2}}]_{\bar{n}} - [e_1, e_{\bar{2}}]_p\epsilon^p{}_{\bar{m}}) &= 0\,. \qquad (12.55)
\end{aligned}$$

Note that the above conditions together with those in (12.47) imply that $\partial_2\Phi = 0$. Therefore the dilaton is invariant under all parallel vector fields e_a. Moreover $\tilde{d}e^a$, $a = -, +, 1$, take values in $\mathfrak{u}(2)$, and the hermitian traces depend on the structure constants of $\mathfrak{h}$. Similarly the anti-self dual part of $\tilde{d}e^2$ depends on the structure constants of $\mathfrak{h}$. So again there is supersymmetry enhancement to $N = 8$, if $\mathfrak{h}$ is abelian.

12.7 N=6

12.7.1 Killing spinors

As in the $N = 5$ case, we use the gauge symmetry to determine the normals to the Killing spinors. Comparing with the $N = 2$ case, we conclude that there are two different possibilities.

12.7.2 $\mathcal{A}1 = \mathcal{A}e_{15} = \mathcal{A}e_{12} = 0$

The solution of the dilatino Killing spinor equation is

$$\begin{aligned}
\partial_-\Phi = \partial_+\Phi = \partial_1\Phi = \partial_2\Phi &= 0\,,\\
H_{+-\bar{1}} - H_{\bar{1}2\bar{2}} = H_{+-\bar{2}} - H_{\bar{2}1\bar{1}} = H_{+1\bar{1}} + H_{+2\bar{2}} = H_{+12} = H_{-1\bar{2}} &= 0\,,\\
[e_1, e_2]_{\bar{m}} = [e_1, e_2]_m = [e_1, e_{\bar{2}}]_{\bar{m}} = [e_+, e_-]_{\bar{n}} - [e_1, e_{\bar{1}}]_{\bar{m}} = [e_1, e_{\bar{1}}]_{\bar{m}} + [e_2, e_{\bar{2}}]_{\bar{m}} &= 0\,,\\
[e_+, e_1]_m = [e_+, e_2]_m = [e_+, e_1]_{\bar{m}} = [e_+, e_2]_{\bar{m}} &= 0\,,\\
[e_-, e_1]_{\bar{m}} = [e_-, e_2]_m &= 0\,,\\
(de_+)_n{}^n = (de_+)_{mn} = (de_-)_n{}^n - H_{-1\bar{1}} + H_{-2\bar{2}} = (de_-)_{mn} &= 0\,,\\
(de_1)_n{}^n = (de_1)_{mn} = (de_1)_{\bar{m}\bar{n}} &= 0\,,\\
(de_2)_n{}^n = (de_2)_{mn} = (de_2)_{\bar{m}\bar{n}} &= 0\,,\\
\partial_{\bar{n}}\Phi - \frac{1}{2}(\theta_{\omega_I})_{\bar{n}} + [e_-, e_+]_{\bar{n}} &= 0\,. \qquad (12.56)
\end{aligned}$$

Clearly the dilaton is invariant under all parallel vectors e_a. Moreover $\tilde{d}e^a$, $a = -, 1, 2$ take values in $\mathfrak{su}(2)$, while $\tilde{d}e^+$ takes values in $\mathfrak{u}(2)$. The hermitian trace of the latter is determined by the structure constant of $\mathfrak{h}$. So there is supersymmetry enhancement to $N = 8$, if $\mathfrak{h}$ is abelian.

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12.7.3 $\mathcal{A}1 = \mathcal{A}e_{15} = \mathcal{A}(e_{25} - e_{1345}) = \mathcal{A}(e_{12} + e_{34}) = 0$

The solution of this Killing spinor equation is given in (12.54) and supplemented with the conditions

$$\begin{aligned} (de_-)_n{}^n = (de_1)_n{}^n - (de_{\bar{1}})_n{}^n &= 0\,, \\ -[e_-, e_{\bar{2}}]_m + [e_-, e_{\bar{1}}]_{\bar{p}}\epsilon^{\bar{p}}{}_m = -[e_1, e_{\bar{1}}]_{\bar{n}} + [e_2, e_{\bar{2}}]_{\bar{n}} - [e_1, e_{\bar{2}}]_p\epsilon^p{}_{\bar{m}} &= 0\,. \end{aligned} \tag{12.57}$$

The dilaton is invariant under all parallel vector fields e_a. Moreover $\tilde{de}^a$, $a = -, +$, take values in $\mathfrak{su}(2)$, and $\tilde{de}^1$ takes values in $\mathfrak{u}(2)$ with the hermitian trace to depend on the structure constants of $\mathfrak{h}$. Similarly the anti-self dual part of $\tilde{de}^2$ depends on the structure constants of $\mathfrak{h}$. So again there is supersymmetry enhancement to $N = 8$, if $\mathfrak{h}$ is abelian.

12.8 N=7

12.8.1 $\mathcal{A}1 = \mathcal{A}e_{15} = \mathcal{A}e_{12} = \mathcal{A}(e_{25} - e_{1345}) = 0$

The solution is given by (12.56) with the additional constraints

$$(de_-)_m{}^m = 0\,, \quad [e_-, e_2]_{\bar{m}} - [e_-, e_1]_p\epsilon^p{}_{\bar{m}} = [e_-, e_+]_{\bar{m}} - \tfrac{1}{2}[e_1, e_{\bar{2}}]_p\epsilon^p{}_{\bar{m}} = 0\,, \tag{12.58}$$

It is straightforward to see that the difference between the solution of the dilatino Killing spinor equation for $N = 7$ backgrounds and that of $N = 8$ backgrounds, see [28] and (12.59) below, is that in the former case the commutators $[e_-, e_2]_i$ and $[e_-, e_+]_i$ do not vanish. Therefore if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, then the $N = 7$ backgrounds admit eight supersymmetries. We will discuss the case $[\mathfrak{h}, \mathfrak{h}] \not\subseteq \mathfrak{h}$ in section 12.10.

12.9 Comparison with N=8

The solutions to the dilatino Killing spinor equation are [28]

$$\begin{aligned} \partial_a\Phi &= 0\,, \qquad (de_a)_n{}^n = 0\,, \qquad (de_a)_{mn} = 0\,, \\ [e_a, e_b]_i &= 0\,, \quad H_{a_1a_2a_3} + \frac{1}{3!}\epsilon_{a_1a_2a_3}{}^{b_1b_2b_3}H_{b_1b_2b_3} = 0\,, \\ 2\partial_{\bar{n}}\Phi - (\theta_{\omega_I})_{\bar{n}} &= 0\,, \end{aligned} \tag{12.59}$$

where $\epsilon_{+-1\bar{1}2\bar{2}} = -1$. In particular $\mathfrak{h} = \mathbb{R} < e_a >$, $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ and spans a self-dual Lorentzian Lie algebra. As has been already mentioned, these have been classified in [45].

It is clear that the differences between $N = 8$ and $1 \leq N < 8$ supersymmetric backgrounds lie in the invariance properties of the dilaton under the action of the parallel vector field, the properties of the commutator $[\mathfrak{h}, \mathfrak{h}]$ and the values of the rotations $\tilde{de}^a$ in $\mathfrak{su}(2)^\perp \subset \Lambda^2(\mathbb{R}^4)$. All the different cases can be characterized in terms of these three criteria. However unlike the cases with parallel spinors that admit a non-compact isotropy group, the comparison is much more involved. In the case by case analysis we have presented, we have not used the fact that there is a classification of Lorentzian metric Lie groups. In particular, this may impose some additional conditions on the structure constants of $\mathfrak{h}$ that arise from the Jacobi identities. In turn, this may lead to some additional simplifications to the solutions of the dilatino Killing spinor equations. We shall investigate this aspect elsewhere.

12.10 Reduction of holonomy

As in the investigation of the holonomy reduction in previous cases, we assume that $dH = 0$, $\mathrm{hol}(\hat\nabla) \subseteq \mathrm{SU}(2)$ and use the field equations to identify the additional $\hat\nabla$-parallel forms. As before either $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ or $\mathrm{hol}(\hat\nabla) \subset \mathrm{SU}(2)$. Similarly, one can show that the H_{abc} are constant and they can be identified with the structure constants of $\mathfrak{h}$.

In fact, in this case the constraints that arise from $[\mathfrak{h}, \mathfrak{h}] \not\subseteq \mathfrak{h}$ are particularly strong. Suppose there is some component $H_{abi} = -[X_a, X_b]_i$ non-vanishing. Since it is parallel with respect to $\hat\nabla$ the supercurvature has to satisfy the integrability condition

$$\hat R_{AB,i}{}^j H_{abj} = 0 \,, \tag{12.60}$$

One can show that this implies the vanishing of $\hat R$ in the following way. First suppose that either H_{ab3} or H_{ab4} vanishes. Then the above constraint readily implies the vanishing of the supercurvature. If both components of H_{abi} are non-vanishing, one can show that the supercurvature has to satisfy

$$-\hat R_{AB,3\bar4}\hat R_{AB,\bar34} = -\hat R_{AB,3\bar3}\hat R_{AB,\bar44} \,. \tag{12.61}$$

The left hand side is non-positive while the right hand side is non-negative (using the fact that $\hat R$ takes values in SU(2)) and hence the supercurvature has to vanish. We conclude that in the SU(2) case either $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ or $\mathrm{hol}(\hat\nabla) = 1$.

Furthermore,

$$\tau_1 = iH_{ap}{}^p\, e^a \tag{12.62}$$

is $\hat\nabla$-parallel. Since e^a are also $\hat\nabla$-parallel $iH_{ap}{}^p = u_a$ are constants. Similarly one can show that

$$\tau_2^a = \frac{1}{2} H^a{}_{pq} e^p \wedge e^q \,, \tag{12.63}$$

are also $\hat\nabla$-parallel. In this case, one can set

$$\tau_2^a = \lambda^a\, \omega_J^{2,0} \,, \quad \lambda^a \in \mathbb{C} \,. \tag{12.64}$$

Using similar arguments to those we have made for the G_2 case, one can also show that

$$\tau_3 = \partial_a \Phi\, e^a \tag{12.65}$$

and

$$\tau_4 = (2\partial_i \Phi - (\theta_{\omega_I})_i)\, e^i \tag{12.66}$$

are $\hat\nabla$-parallel. Since e^a are also $\hat\nabla$-parallel, $\partial_a \Phi = v_a$ are constants. Similarly, either $\tau_4 = 0$ or $\mathrm{hol}(\hat\nabla) \subset \mathrm{SU}(2)$.

Let us now turn to investigate some of the implications that the above parallel forms have for supersymmetric backgrounds. As can be seen from the conditions for $N = 8$

backgrounds, $dH = 0$, $\mathrm{hol}(\hat{\nabla}) = \mathrm{SU}(2)$ and the field equations are not sufficient to imply the dilatino Killing spinor equations from the gravitino ones. In particular, one has to impose in addition $\tau_1 = \tau_2 = \tau_3 = 0$. Moreover, a direct inspection of the conditions for the descendants with $N < 7$ reveals that they may be solutions with $\mathrm{hol}(\hat{\nabla}) = \mathrm{SU}(2)$. All such solutions are principal bundles over a four-dimensional manifold. However the base manifold may not admit an SU(2)-structure.

For $N = 7$ we found that if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$ this would reduce to the $N = 8$ case. However, due to the above integrability condition (12.60), if $[\mathfrak{h}, \mathfrak{h}] \nsubseteq \mathfrak{h}$ the holonomy of $\hat{\nabla}$ reduces to the identity. As mentioned before and as will be discussed in section 13, such backgrounds preserve at least 8 supersymmetries. This arises as a consequence of the conditions $dH = \hat{R} = 0$ and the dilatino Killing spinor equation [32, 31]. This case is reminiscent of type II backgrounds with 31 supersymmetries [34 – 36].

13. The descendants of 1

The Killing spinor equation implies that $\hat{R} = 0$ and so the spacetime is parallelizable. In addition one can argue that $dH = 0$. This is certainly the case in the lowest order in α'. The Bianchi identity of H receives anomaly contributions from the gravitational sector and the gauge sector. The gravitational contribution can be expressed in terms of $\check{R}$ where $\check{R}$ can be found from $\hat{R}$ after setting H to $-H$. Now if $dH = 0$, $\hat{R}_{AB,CD} = \check{R}_{CD,AB}$ and since $\hat{R} = 0$ for these backgrounds, the gravitational contribution to the anomaly vanishes. The gauge contribution also vanishes if we assume that all parallel spinors also solve the gaugino Killing spinor equation. Parallelizable supersymmetric backgrounds with $dH \neq 0$ have been investigated in [31].

The dilatino Killing spinor equation imposes additional conditions on the spacetime. There are two cases to consider depending on whether or not the one-form $d\Phi$ is null. Suppose that $d\Phi$ is not null and $|d\Phi|^2 \neq 0$. In this case, one can show that the dilatino Killing spinor equation [32, 31] implies that

$$\Pi = \frac{1}{2} + \frac{\partial_M \Phi H_{NPQ} \Gamma^{MNPQ}}{24 |d\Phi|^2} \,, \tag{13.1}$$

is a projector, $\Pi^2 = \Pi$. Since $\mathrm{tr}\,\Pi = 8$, backgrounds with $dH = \hat{R} = 0$ and $|d\Phi| \neq 0$ preserve at least half of the supersymmetry. Moreover one can also show that $d\Phi$ is ∇-parallel, spacelike and $i_{d\Phi} H = 0$, see e.g [32, 28]. All these are linear dilaton backgrounds. Moreover $d\Phi$ spans a flat direction orthogonal to the rest of spacetime.

On the other hand if $|d\Phi| = 0$, i.e. either $d\Phi$ is null or $d\Phi = 0$, then H is null. If $d\Phi \neq 0$, then using the condition $i_{d\Phi} H = 0$ one can show that these backgrounds preserve at least eight supersymmetries. In the following we confirm the results of [31].

13.1 N=8

The solutions for which $|d\Phi| \neq 0$ have been classified and have been found to be isometric to

$$AdS_3 \times S^3 \times S^3 \times \mathbb{R}\,, \quad AdS_3 \times S^3 \times \mathbb{R}^4\,, \quad \mathbb{R}^{1,1} \times \mathrm{SU}(3)\,, \quad \mathbb{R}^{3,1} \times S^3 \times S^3\,,$$
$$\mathbb{R}^{6,1} \times S^3\,, \quad CW_4 \times S^3 \times \mathbb{R}^3\,, \quad CW_6 \times S^3 \times \mathbb{R}\,, \tag{13.2}$$

where CW stands for Cahen-Wallach spaces. In fact it turns out that the full content of the dilatino Killing spinor equation is the projection $\Pi\epsilon = \epsilon$. So these backgrounds preserve precisely 8 supersymmetries.

On the other hand if $|d\Phi| = 0$, $d\Phi \neq 0$, it has been shown that the spacetime is isometric to

$$CW_{10}\,, \quad CW_8 \times \mathbb{R}^2\,, \quad CW_6 \times \mathbb{R}^4\,, \quad CW_4 \times \mathbb{R}^6\,, \quad \mathbb{R}^{9,1}\;. \tag{13.3}$$

The eight Killing spinors can be chosen, up to a gauge transformation, to satisfy $\Gamma^+\epsilon = 0$. A basis in the space of these Killing spinors is $(e_{\alpha 5}, e_{\alpha\beta\gamma 5})$, i.e. these Killing spinors span the $\Delta^-_{\mathbf{8}}$ representation of Spin(8). All these backgrounds can be thought of as degenerations of CW_{10}. The only non-vanishing component of the flux is

$$H = e^+ \wedge \beta\,, \quad \beta = \frac{1}{2}\beta_{ij} e^i \wedge e^j\,, \tag{13.4}$$

and the dilaton is linear. The form β is a generic element in $\Lambda^2(\mathbb{R}^8) = \mathfrak{spin}(8)$. The Maurer-Cartan structure equations of the Cahen-Wallach group manifolds are

$$de^+ = 0\,, \quad de^- = \frac{1}{2}\beta_{ij} e^i \wedge e^j\,, \quad de^i = -\beta^i{}_j e^+ \wedge e^j\;. \tag{13.5}$$

The backgrounds with $N > 8$ supersymmetries are special cases of such backgrounds with constant dilaton and appropriate restrictions on β.

13.2 N= 10

To begin observe that $\Sigma(\mathcal{K}) = \mathrm{Spin}(1,1) \times \mathrm{Spin}(8)$. The additional Killing spinor lies in $\Delta^+_{\mathbf{8}}$. Using a similar argument to that we have applied to determine the descendants in the $\mathbb{R}^8$ case, the Killing spinor can be chosen as

$$\epsilon_9 = 1 + e_{1234}\;. \tag{13.6}$$

The dilatino Killing spinor equation gives

$$\partial_+\Phi = 0, \qquad H_{+\alpha}{}^{\alpha} = 0, \qquad H_{+\alpha\beta} - \tfrac{1}{2}\epsilon_{\alpha\beta}{}^{\bar\gamma\bar\delta} H_{+\bar\gamma\bar\delta} = 0\;., \tag{13.7}$$

Therefore for such backgrounds, we have found that

$$\Phi = \mathrm{const}\,, \quad \beta \in \mathfrak{spin}(7)\;. \tag{13.8}$$

Since β is constant and Spin(7) acts on the remaining spinors with the vector representation, the dilatino Killing spinor equation $\beta_{ij}\Gamma^{ij}\epsilon = 0$ has another solution. This is because an element of $\mathrm{SO}(2k+1)$ acting of $\mathbb{R}^{2k+1}$ leaves an axis invariant. The stability subgroup in this case is $SU(4) = Spin(6)$. Therefore, there is supersymmetry enhancement to $N = 10$. The second spinor can be chosen as

$$\epsilon_{10} = i(1 - e_{1234}) \tag{13.9}$$

and $\beta \in \mathfrak{su}(4)$. These backgrounds are the special cases of Cahen-Wallach space-times that have two additional supersymmetries [31]. Therefore there are no *isolated* backgrounds with $N = 9$ supersymmetry. However deformations families of $N = 10$ backgrounds which can be constructed by allowing β to take values in $\mathfrak{spin}(7)$ have $N = 9$ supersymmetries.

13.3 N=12

Applying the same arguments as in the $\mathbb{R}^8$ case, we can choose an additional solution of the dilatino Killing spinor equation $\beta_{ij}\Gamma^{ij}\epsilon = 0$ as $\epsilon_{11} = i(e_{12} + e_{34})$. The stability subgroup is $Sp(2) = Spin(5)$. Again $Sp(2)$ acts with the vector representation on the remaining spinors and so there is an additional Killing spinor which can be chosen as $\epsilon_{12} = e_{12} - e_{34}$ with stability subgroup $Spin(4)$ and so

$$\beta \in \mathfrak{spin}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \ . \tag{13.10}$$

Again there are no backgrounds with $N = 11$ supersymmetries. Moreover the components of β for $N > 8$ are those of H_{+ij} found for the corresponding descendants of the $\mathbb{R}^8$.

13.4 N=14

It can be arranged such that the next solution of the dilatino Killing spinor equation is $\epsilon_{13} = e_{13} + e_{24}$ with stability subgroup $Spin(3)$ which again acts on the remaining three spinors with the vector representation. Therefore there is an addition Killing spinor which can be chosen $\epsilon_{14} = i(e_{13} - e_{24})$. Moreover

$$\beta \in \mathfrak{su}(2) \ . \tag{13.11}$$

13.5 N=16

There no backgrounds with $N = 15$ supersymmetries, see e.g [35]. The backgrounds with $N = 16$ supersymmetries are isometric to Minkowski spacetime [37].

14. Concluding remarks

We have solved, using the spinorial geometry technique of [29], the Killing spinor equations for all supersymmetric type I backgrounds. In particular, we utilized the gauge symmetry of the Killing spinor equations of the theory to construct representatives for the Killing spinors in all cases. We have approached the problem by first solving the gravitino Killing spinor equation whose solutions are parallel spinors with respect to a metric connection, $\hat{\nabla}$, with torsion a three-form H. The solutions have been characterized by the isotropy group of the spinors in $Spin(9,1)$. Then in each case, we have used as gauge symmetry the subgroup $\Sigma(\mathcal{P})$ of $Spin(9,1)$ that leaves invariant the space of parallel spinors $\mathcal{P}$ to find representatives for the solutions of the dilatino Killing spinor equation for the descendant backgrounds. The Killing spinors are characterized by the isotropy group of the associated parallel spinors and their stability subgroup in $\Sigma(\mathcal{P})$.

There are two classes of supersymmetric backgrounds depending on whether the isotropy group of the parallel spinors is compact K or non-compact $K \ltimes \mathbb{R}^8$. In the latter case, all backgrounds admit a null $\hat{\nabla}$-parallel vector field. Moreover, their geometries can be characterized in terms of the properties of the rotation of the parallel vector field and those of endomorphisms of the tangent bundle that behave as almost complex structures on the transverse directions to the light-cone. In particular, the geometries depend on the integrability of these endomorphisms and on a relation between their Lee forms. In addition the Gray-Hervella class W_2 vanishes for all the endomorphisms. We have also shown that if one imposes $dH = 0$ and the field equations, then the holonomy of $\hat{\nabla}$ reduces for all the descendant backgrounds. This is because the spacetime admits more parallel forms than those allowed by the holonomy of $\hat{\nabla}$. Moreover, under the same assumptions, if one insists that the holonomy of $\hat{\nabla}$ is precisely the isotropy group of the parallel spinors, then the gravitino Killing spinor equation implies the dilatino one and all parallel spinors are Killing. These are the backgrounds explored in [28].

On the other hand if the isotropy group of the parallel spinors is compact, i.e. G_2, SU(3), SU(2) and $\{1\}$, then the spacetime admits, 3, 4, 6 and 10 $\hat{\nabla}$-parallel vector fields, respectively. In addition all the invariant forms associated with these groups are also $\hat{\nabla}$-parallel. The geometry of the backgrounds depends on the properties of the rotation of the parallel vector fields and their commutators, the integrability conditions of the endomorphisms invariant under the above groups, and the relation between the Lee forms of the remaining $\hat{\nabla}$-parallel forms. In addition $W_2 = 0$ vanishes for all endomorphisms associated with the invariant Hermitian forms. The pattern of relations between the various tensors that characterize the geometry is more involved in this case. We have also shown that if $dH = 0$ and the field equations are satisfied, then in many cases there are additional parallel forms on the spacetime than those allowed by the holonomy groups. Therefore if these additional forms do not vanish, the holonomy reduces. Hence, if one insists that the holonomy of $\hat{\nabla}$ is precisely the isotropy group of the parallel spinors, this imposes additional conditions on the existence of descendants. In particular, this would imply that the vector space spanned by $\hat{\nabla}$-parallel vector fields closes under Lie brackets and many terms in the solution of dilatino Killing spinor equation for the descendants would vanish. However unlike the non-compact case, there are descendants with holonomy precisely the isotropy group of the parallel spinors.

As we have already mentioned, the assumption that the holonomy of $\hat{\nabla}$ is precisely the isotropy group of the parallel spinors puts strong conditions on the existence of most descendants. However, one may allow the holonomy of $\hat{\nabla}$ to be reduced. In some of the cases, this will imply enhancement of supersymmetry but not always. We have shown that the $N = 7$ descendant of SU(2) always admits an additional supersymmetry and so it can be identified with the $N = 8$ backgrounds. The full pattern or web of reductions is rather involved and it may be worth a systematic investigation. There are additional conditions on the geometry that we have not investigated. For example, there is a classification of

Lorentzian Lie groups [33] and so a priori there are additional conditions on the structure constants of the Lie algebra of the parallel vectors. These have not been implemented in the analysis of the descendants. This mostly affects the descendants of the SU(2) case and the results will be reported elsewhere [46].

So far we have investigated the geometry of supersymmetric backgrounds. A natural question arises whether all solutions can be classified. If a background admits Killing spinors with a non-compact isotropy group, then from the results of [28], $dH = 0$, the Killing spinor equations, and the vanishing of E_{--} and LH_{-+} components of the Einstein and two-form gauge potential field equations, respectively, imply that all field equations are satisfied. Similarly, if a background admits Killing spinors with a compact isotropy group, then the Killing spinor equations and $dH = 0$ imply all field equations. However despite these simplifications, it is unlikely that all the solutions can be classified in full generality in the near future. This is because such a task is related to other classic classification problems like for example those of G_2 and Spin(7) manifolds that remain unresolved. Nevertheless some classes of solutions can be understood better. One such class is that of compactifications of type I supergravities with fluxes. It is clear that some backgrounds with N, $N = 1, 2, 3, 4, 5, 6, 8$, parallel spinors which have a non-compact isotropy group can serve as the vacuum configurations of compactification of type I to $1 + 1$ dimensions. This is confirmed by the property of $\Sigma(\mathcal{P})$ to be isomorphic to $\mathrm{Spin}(1, 1) \times R$ where R and be thought of as an R-symmetry group of the $1 + 1$ supergravity. A similar observation can be made for backgrounds with parallel spinors which have compact isotropy groups. In particular backgrounds with parallel spinors that have G_2, SU(3) and SU(2) isotropy groups can be used for compactifications to $2 + 1$-, $3 + 1$- and $5 + 1$-dimensions. The $\Sigma(\mathcal{P})$ group has the appropriate structure. It is also possible to go beyond the vacuum configurations and compare supersymmetric solutions of type I supergravity with those of lower dimensional supergravities that are related via a compactification. This will give an insight into how supersymmetric solutions are related in a compactification scenario.

One may also wonder whether the classification of geometries of all supersymmetric backgrounds in type I supergravity can be extended to those of type II supergravities. The nature of the problem in type II is different. This is because the gauge group of the Killing spinor equations in type II supergravities is a proper subgroup of the holonomy group of the supercovariant connection. This, and its consequences, have been explained in detail in the conclusions of [44] and we shall not repeat the analysis here. Nevertheless the results of this paper can be adapted to solve the algebraic Killing spinor equations of type II supergravities provided that a solution of the gravitino Killing spinor equation is known. In particular the group $\Sigma(\mathcal{P})$ that preserves the space of parallel spinors can again be introduced and then it can be used to find representatives for the solutions of the algebraic Killing spinor equations. Clearly, this can be applied in type IIA and IIB supergravities and well as in other supergravities in lower dimensions.

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A. Geometric structures

A.1 Compact stability subgroup

To determine the geometry of supersymmetric backgrounds, one has to understand the different geometric structures that can occur. For parallel spinors with compact stability subgroups K in $\mathrm{Spin}(9,1)$, and so $\mathrm{hol}(\hat{\nabla}) \subseteq K$, the spacetime admits one time-like and $n = 2$ or 3 or 5 or 9 spacelike $\hat{\nabla}$-parallel vectors fields denoted by X_a and some K-invariant $\hat{\nabla}$-parallel forms, which we denote collectively by τ. It is always possible to choose a basis in the ring of invariant forms such that

$$i_a \tau = 0\,, \tag{A.1}$$

where i_a denotes inner derivation with respect to the vector field X_a. Since X_a are parallel they are nowhere zero and so span a topologically trivial subbundle Ξ of the tangent vector bundle TM of the spacetime M. Thus we have

$$0 \to \Xi \to TM \to \Pi \to 0\,, \tag{A.2}$$

such that Π is the orthogonal complement of Ξ in TM with respect to the spacetime metric, $TM = \Xi \oplus \Pi$. Since Ξ is trivial, the topological structure group of M reduces to $K \subset \mathrm{Spin}(9-n) \subset \mathrm{Spin}(9,1)$. In general $[\Xi, \Xi] \nsubseteq \Xi$, and so M is *not* always foliated. Nevertheless, the above decomposition of TM and its dual can be used to decompose the various tensors of M along the directions of Ξ and Π. In particular, introduce the dual one forms e^a of X_b, i.e. $e^a(X_b) = \delta^a{}_b$. Since $\hat{\nabla}$ is a metric connection $g(X_a, X_b)$ is constant and so one can always choose $g(X_a, X_b) = \eta_{ab}$, where η_{ab} is the standard Lorentz metric. Therefore, we can set for the metric and H,

$$\begin{aligned} ds^2 &= \eta_{ab} e^a e^b + \delta_{ij} e^i e^j\,, \\ H &= \frac{1}{3!} H_{abc} e^a \wedge e^b \wedge e^c + \frac{1}{2} H_{abi} e^a \wedge e^b \wedge e^i + \frac{1}{2} H_{aij} e^a \wedge e^i \wedge e^j + \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k\,, \end{aligned} \tag{A.3}$$

where e^i is a local basis of one-forms spanning the fibers of the dual of Π, i.e. the spacetime frame index decomposes as $A = (a, i)$ and $H|_\Pi = \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k$. In this basis, the remaining $\hat{\nabla}$-parallel forms can be written as

$$\tau = \frac{1}{k!} \tau_{i_1,\dots,i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}\,, \tag{A.4}$$

i.e. $\tau = \tau|_\Pi$. This follows from (A.1).

To find the conditions imposed on the geometry by $\hat{\nabla} X_a = \hat{\nabla}\tau = 0$, we observe that

$$\begin{aligned} &\hat{\nabla}_A (X_a)_B = 0 \Longleftrightarrow i_a H = \eta_{ab} de^b\,, \quad \mathcal{L}_a g = 0\,, \\ &\hat{\nabla}_A \tau_{B_1\dots B_k} = 0 \Longleftrightarrow \nabla_a \tau_{j_1\dots j_k} = \frac{k}{2}(-1)^k H_a{}^i{}_{[j_1} \tau_{j_2\dots j_k]i}\,, \quad \hat{\nabla}_i \tau_{j_1\dots j_k} = 0\,. \end{aligned} \tag{A.5}$$

Therefore X_a is Killing and its rotation is given in terms of H. In turn this implies that all components of H of the type H_{aAB} are determined. In particular, we have that

$$\begin{aligned} &H_{aij} = (i_a H)_{ij} = (de_a)_{ij}\,, \qquad H_{abi} = (i_b i_a H)_i = (de_a)_{bi} = -[X_a, X_b]_i\,, \\ &H_{abc} = i_c i_b i_a H = (de_a)_{bc} = -g([X_a, X_b], X_c)\,, \end{aligned} \tag{A.6}$$

where $de_a = \eta_{ab}(de^b)$. If $[\Xi, \Xi] \subseteq \Xi$, i.e. X_a span a Lie algebra, then $H_{abi} = 0$. We shall examine this case in more detail later.

Next focus on the conditions in (A.5) involving τ. It is clear from the above equations that some of the components H_{aij} are also determined in terms the the covariant derivative of τ. Compatibility requires a restriction on the geometry, i.e. a relation between the exterior derivative of X_a, which also determines H_{aij}, and the covariant derivative of τ. To find such geometric conditions, we begin with the first pair of the above equations, and decompose $\Lambda^2(\mathbb{R}^{9-n}) = \mathfrak{k} \oplus \mathfrak{k}^\perp$, where $\mathfrak{k}$ is the Lie algebra of K. This induces a decomposition of the two-form $i_a H|_\Pi$ as $i_a H|_\Pi = i_a H^{\mathfrak{k}} + i_a H^{\mathfrak{k}^\perp}$. It is clear that $i_a H^{\mathfrak{k}}$ is not determined by the first equation because the forms τ are invariant under the action of K. However, $i_a H^{\mathfrak{k}^\perp}$ can be expressed in terms of both the covariant derivative of τ and the rotation of X_a. As a result, the $\mathfrak{k}^\perp$ component of the rotation of X_a can be expressed in terms of the ∇_a covariant derivative of τ, i.e schematically we have

$$(de_a)^{\mathfrak{k}^\perp} = (\nabla_a \tau)^{\mathfrak{k}^\perp}\,. \tag{A.7}$$

It remains to investigate the condition $\hat\nabla_i \tau_{j_1\dots j_k} = 0$. This condition can be used to investigate the $H|_\Pi$ component of H. The analysis is similar to that which one does in the context of $(9-n)$-dimensional manifolds with K-structure compatible with a connection with skew-symmetric torsion. The end results depends on the K structure, it may or may not give additional conditions on the geometry. In all cases, $H|_\Pi$ is entirely determined in terms of the geometry. We shall not give further details here but we describe the end result in each case separately.

Using $\hat\nabla X_a = \hat\nabla \tau = 0$, one can also compute the Lie derivative of τ along X_a to find

$$\begin{aligned} \mathcal{L}_a \tau_{A_1 A_2 \dots A_k} = k(-1)^k H_a{}^B{}_{[A_1} \tau_{A_2\dots A_k]B} \iff \mathcal{L}_a \tau_{i_1 i_2 \dots i_k} &= k(-1)^k H_a{}^j{}_{[i_1} \tau_{i_2\dots i_k]j}\,, \\ \mathcal{L}_a \tau_{b i_1 \dots i_{k-1}} &= (-1)^k H_a{}^j{}_b \tau_{i_1\dots i_{k-1} j}\,. \end{aligned} \tag{A.8}$$

Thus if $i_a H^{\mathfrak{k}^\perp}$ vanishes and $[\Xi, \Xi] \subseteq \Xi$, then $\mathcal{L}_a \tau = 0$. Moreover observe that if $dH = 0$, then $\mathcal{L}_a H = 0$.

One can utilize the relation of H to the rotation of X_a to write H in terms of X_a in various ways. For example, one can write

$$\begin{aligned} H = &\,\eta_{ab}\, e^a \wedge de^b + \frac{1}{3} g([X_a, X_b], X_c) e^a \wedge e^b \wedge e^c + \frac{1}{2}[X_a, X_b]_i e^a \wedge e^b \wedge e^i \\ &+ \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k\,, \end{aligned} \tag{A.9}$$

where as we have mentioned the expression for $H|_\Pi$ depends on the K-structure.

As has been observed in [28], there is an alternative way to write H in the case that $[\Xi, \Xi] \subseteq \Xi$. In particular, one has that $H_{abi} = 0$, the spacetime is a principal bundle, $\lambda^a = e^a$

is identified with a principal bundle connection, and H_{abc} are the structure constants of the Lie algebra spanned by X_a. In this case, it is more convenient to write

$$H = \frac{1}{3}\eta_{ab}\lambda^a \wedge d\lambda^b + \frac{2}{3}\eta_{ab}\lambda^a \wedge \mathcal{F}^b + \frac{1}{3!}H_{ijk}e^i \wedge e^j \wedge e^k \,, \tag{A.10}$$

where

$$\mathcal{F} = \frac{1}{2}H^a{}_{ij}e^i \wedge e^j = d\lambda^a - \frac{1}{2}H^a{}_{bc}\lambda^b \wedge \lambda^c \,, \tag{A.11}$$

is the curvature of the principal bundle. Sometimes we write $H^{\rm rest} = H|_\Pi$.

The dilatino Killing spinor equation will impose additional conditions on H and on the geometry. These are determined on a case by case basis from the solutions of the dilatino Killing spinor equations and depend on the choice of Killing spinors up to Lorentz transformations. This is unlike the conditions we have described above which depend on the $\hat\nabla$-parallel spinors that the spacetime admits, i.e. the solutions of the gravitino Killing spinor equation.

A.2 Non-compact stability subgroup

If the stability subgroup of the parallel spinors is not compact, $K \ltimes \mathbb{R}^8$, the spacetime admits a $\hat\nabla$-parallel null vector field X and null $\hat\nabla$-parallel forms which we collectively denote with τ such that

$$i_X\tau = 0 \;. \tag{A.12}$$

Since the null vector field is nowhere vanishing, the tangent bundle of the spacetime admits a trivial rank one subbundle Ξ and so

$$0 \rightarrow \Xi \rightarrow TM \rightarrow L \rightarrow 0 \;. \tag{A.13}$$

Choosing $X = e_+$, and so the associated $\hat\nabla$-parallel one-form is e^-, the spacetime metric can be written as

$$ds^2 = 2e^-e^+ + \delta_{ij}e^ie^j \,, \tag{A.14}$$

where e^+, e^i is a local basis in L. The structure group of TM, which is a subgroup of the holonomy group $K \ltimes \mathbb{R}^8 \subset \mathrm{Spin}(8) \ltimes \mathbb{R}^8$, acts as

$$e^- \rightarrow e^- \,, \quad e^+ \rightarrow e^+ - O_{ij}q^i\, e^j - O_{ij}q^iq^j\, e^- \,, \quad e^i \rightarrow O^i{}_j\, e^j + q^i\, e^- \,, \tag{A.15}$$

where O is an element of the vector representation of Spin(8) and $q \in \mathbb{R}^8$. There is no natural definition of the e^+ light-cone direction or of the “transverse” e^i directions to the lightcone. Next observe the bundle of $(k+1)$-forms of M, $\Lambda^{k+1}(M)$, contains a subbundle

$$N^{k+1} = \{\alpha \in \Lambda^{k+1}(M)\,, \text{s.t.,}\;\; i_X\alpha = 0\,, \quad e^- \wedge \alpha = 0\} \;. \tag{A.16}$$

The $\hat\nabla$-parallel forms τ are sections of this bundle. The transition functions of N^{k+1} are those associated with the k-fold skew-symmetric product of the vector representation of

SO(8), i.e. the transition functions of N^{k+1} are those of a k-form bundle of a "transverse space" to the light-cone. In particular, one can define "transverse" $(k+1)$-forms on the spacetime M up to sections of N^{k+1}. This can be seen from the sequence

$$0 \to N^{k+1} \to M^{k+1} \to \Omega^{k+1} \to 0 \tag{A.17}$$

where

$$M^{k+1} = \{\alpha \in \Lambda^{k+1}(M)\,, \text{s.t.,}\ i_X\alpha = 0\ \}\,, \tag{A.18}$$

and the "transverse forms" are sections of Ω^{k+1}. Moreover observe that the map $e^-\wedge : \Omega^k \to N^{k+1}$ is an isomorphism. In addition Ω^{k+1} is equipped with a fiber metric induced from the spacetime metric.

The $\hat\nabla$-parallel forms τ can be written as $\tau = e^- \wedge \phi$, where ϕ are K-invariant forms which can be thought of as sections of Ω^k. The condition that X and τ are $\hat\nabla$-parallel can be written as

$$\begin{aligned}
\hat\nabla_A X_B = 0 \Longleftrightarrow\ & de^- = i_X H\,, \quad \mathcal{L}_X g = 0\\
\hat\nabla_A \tau_{B_1\dots B_{k+1}} = 0 \Longleftrightarrow\ & \nabla_+\phi_{j_1\dots j_k} = \frac{k}{2}(-1)^k H_+{}^i{}_{[j_1}\phi_{j_2\dots j_k]i}\,,\\
& \hat\nabla_-\phi_{j_1\dots j_k} = \nabla_-\phi_{j_1\dots j_k} + (-1)^k\frac{k}{2}H^i{}_{-[j_1}\phi_{j_2\dots j_k]i} = 0\\
& \hat\nabla_i\phi_{j_1\dots j_k} = 0\,,
\end{aligned} \tag{A.19}$$

So the $i_X H_{AB} = H_{+AB}$ components of H are determined in terms of e^-, and X is a Killing vector field. Next let us focus on

$$\nabla_+\phi_{j_1\dots j_k} = \frac{k}{2}(-1)^k H_+{}^i{}_{[j_1}\phi_{j_2\dots j_k]i}\ . \tag{A.20}$$

This can be viewed as conditions on H_{+ij}. Since $i_X i_X H = 0$, $i_X H$ is a section of M^2. The above condition imposes a restriction on the "transverse" components of $i_X H$. In particular, decomposing $\Lambda^2(\mathbb{R}^8) = \mathfrak{k} \oplus \mathfrak{k}^\perp$, (A.20) is independent of $i_X H^{\mathfrak{k}}$, and expresses $i_X H^{\mathfrak{k}^\perp}$ in terms of the covariant derivative of τ along the X direction. In turn this is related to the $\mathfrak{k}^\perp$ component of the rotation de^-. This is a condition on the geometry as that of (A.7) for compact stability subgroups mentioned above. Similarly, one can see from the remaining conditions in (A.19) that the $H^{\mathfrak{k}}_-$ is not determined by the parallel transport equation while the $H^{\mathfrak{k}^\perp}_-$ is expressed in terms of the ∇_- covariant derivative of τ.

It remains to investigate the condition $\hat\nabla_i\phi_{j_1\dots j_k} = 0$. This condition can be analyzed as though it is examined in the context of 8-dimensional manifolds with K-structure compatible with a connection with skew-symmetric torsion. This is because as we have mentioned Ω^k has the properties of the bundle of k-forms of "transverse space" to the light-cone. The end result depends on the K structure, it may or may not give additional conditions on the geometry. In all cases, H_{ijk} is entirely determined in terms of the geometry.

The Lorentzian structures we have presented above are reminiscent of the Cauchy-Riemann (CR) structures. This is not a surprise since the CR structures also arise in the context of null Maxwell fields in General Relativity, and they can be associated with a $\mathrm{U}(n) \ltimes \mathbb{R}^{2n}$ type of structures, for a recent review see [43]. One can give various generalizations of the CR structures by using a Gray-Hervella type of classification for the $K \ltimes \mathbb{R}^L$-structures.

The Lie derivative of a k-form along the $\hat{\nabla}$-parallel vector field X is

$$\begin{aligned} \mathcal{L}_X \tau_{A_1 A_2 \dots A_{k+1}} &= (k+1)(-1)^{k+1} i_X H^B{}_{[A_1} \tau_{A_2 \dots A_{k+1}]B} \\ &\Longleftrightarrow \mathcal{L}_X \tau_{-i_1 \dots i_k} = k i_X H^j{}_{[i_1} \tau_{i_2 \dots i_k]j-} \ . \end{aligned} \tag{A.21}$$

Thus $\mathcal{L}_X \tau = 0$ for all τ, iff $i_X H^{\mathfrak{k}^\perp} = 0$.

The geometry and fluxes can be written as

$$\begin{aligned} ds^2 &= 2e^- e^+ + \delta_{ij} e^i e^j \\ H &= e^+ \wedge de^- + \frac{1}{2}(H^{\mathfrak{k}} + H^{\mathfrak{k}^\perp})_{-ij} e^- \wedge e^i \wedge e^j + \frac{1}{3!} H_{ijk} e^i \wedge e^j \wedge e^k \end{aligned} \tag{A.22}$$

where $H^{\mathfrak{k}}_-$ is not determined by the Killing spinor equations.

For pp-wave backgrounds $de^- = 0$. In such a case, one can write $e^- = dv$ for some coordinate v and X is parallel with respect to the Levi-Civita connection. The transverse space B to the pp-wave can then be defined as $u, v =$ const., where u is the affine parameter of the of the null geodesics. In all cases B admits a K-structure, see [28] for more details.

A.3 Integrability conditions, field equations and holonomy

To investigate the existence of certain supersymmetric backgrounds, it is useful to incorporate the Bianchi identities and the field equations in the conditions for supersymmetry. The derivation of the field equations from the integrability conditions of the Killing spinor equations can be found in [41, 28]. Some additional useful formulae are the Bianchi identities

$$\begin{aligned} \hat{R}_{[AB,CD]} &= -\frac{1}{4}(dH)_{ABCD} + \frac{1}{2} H_{E[AB} H^E{}_{CD]} \ , \\ \hat{R}_{A[B,CD]} &= -\frac{1}{3}\hat{\nabla}_A H_{BCD} - \frac{1}{6}(dH)_{ABCD} \ , \\ \hat{R}_{[AB,C]D} &= -\frac{1}{3} dH_{ABCD} - \frac{1}{3}\hat{\nabla}_D H_{ABC} - H^E{}_{D[A} H_{BC]E} \ . \end{aligned} \tag{A.23}$$

of $\hat{R}$. In particular, the second identity will be used to investigate the reduction of the holonomy of $\hat{\nabla}$ for the descendants.

B. Revisiting the singlets

In the introduction, we have listed the Lie subgroups of $\mathrm{Spin}(9,1)$ that leave some Majorana-Weyl spinors invariant. Here we shall provide an argument to show that the list in the introduction is complete. This is essentially a Lie algebra computation. There are two additional cases that occur in the type I backgrounds in addition to those that have not been investigated in [28].

There is a single type of orbit of $\mathrm{Spin}(9,1)$ in Majorana-Weyl representation S^+ of co-dimension zero and a representative is $1 + e_{1234}$, see [27, 28]. Two spinors are invariant either under the subgroup $SU(4) \ltimes \mathbb{R}^8$ or G_2. The representative of the second spinor [28] can be chosen as $i(1 - e_{1234})$ or $e_{15} + e_{2345}$, respectively.

To proceed, we decompose S^+ under the action of SU(4), as[16] $S^+ = (\mathbb{C} < 1 >)_{\mathbb{R}} \oplus \mathrm{Re}\Lambda^2(\mathbb{C}^4) \oplus (\Lambda^1(\mathbb{C}^4))_{\mathbb{R}}$, where we have chosen a 1 as representative for the first two invariant spinors to make the analysis more transparent. There is an orbit of co-dimension one of SU(4) in $\mathrm{Re}\Lambda^2(\mathbb{C}^4)$ with stability stability subgroup Sp(2). In addition under the action of Sp(2), S^+ decomposes as $S^+ = (\mathbb{C} < 1 >)_{\mathbb{R}} \oplus \mathbb{R} < i(e_{12} + e_{34}) > \oplus \Lambda^1(\mathbb{R}^5) \oplus \mathbb{H}^2$, thus there are only three $Sp(2) \ltimes \mathbb{R}^8$-invariant spinors. SU(4) has an orbit of co-dimension 2 in $(\Lambda^1(\mathbb{C}^4))_{\mathbb{R}}$ with stability subgroup SU(3). However this case can be thought of descending from G_2 and so it will be investigated later.

To investigate the case of four invariant spinors, $Sp(2) = Spin(5)$ acts with the vector representation on $\Lambda^1(\mathbb{R}^5) \subset S^+$. So there is a single orbit with stability subgroup $SU(2) \times SU(2)$. In addition under $SU(2) \times SU(2)$, S^+ decomposes as $S^+ = (\mathbb{C} < 1, e_{12} >)_{\mathbb{R}} \oplus \mathbb{H} \oplus (\mathbb{C}^2 \oplus \mathbb{C}^2)_{\mathbb{R}}$, thus there are only four $(SU(2) \times SU(2)) \ltimes \mathbb{R}^8$-invariant spinors. A key point is that $SU(2) \times SU(2)$ acts on $\mathbb{H}$ with the left and right multiplication by unit quaternions, i.e.

$$x \to a x \bar{b}\,, \quad x \in \mathbb{H}\,, \quad a \in SU(2)\,, \quad b \in SU(2)\ . \tag{B.1}$$

Sp(2) has also an orbit in $\mathbb{H}^2$ with stability subgroup Sp(1) but this can also be thought of as the descending from the G_2 case and it will be investigated later.

Next $SU(2) \times SU(2)$ has a single orbit in $\mathbb{H}$ with stability subgroup SU(2), $SU(2) \subset SU(2) \times SU(2)$ is the diagonal subgroup. This case has not been consider in [28]. Moreover under this SU(2), S^+ decomposes as $S^+ = (\mathbb{C} < 1, e_{12} >)_{\mathbb{R}} \oplus \mathbb{R} < e_{13} + e_{24} > \oplus \mathrm{Im}\mathbb{H} \oplus (\mathbb{C}^2)_{\mathbb{R}} \oplus (\mathbb{C}^2)_{\mathbb{R}}$, where SU(2) acts on both copies of $\mathbb{C}^2$ with the fundamental representation. Thus $SU(2) \ltimes \mathbb{R}^8$ leaves invariant five spinors in S^+. Moreover there three types of orbits of $SU(2) \times SU(2)$ in $(\mathbb{C}^2 \oplus \mathbb{C}^2)_{\mathbb{R}}$. Two of those have SU(2) stability subgroup. These two cases can be thought of descending from the G_2 case and they will be investigated later. The third type has trivial stability subgroup and so there are no more invariant spinors.

To proceed, observe that $SU(2) \subset SU(2) \times SU(2)$ acting as (B.1), for $a = b$, on $\mathrm{Im}\mathbb{H}$ has a orbit of codimension one which has stability subgroup U(1). In addition S^+ decomposes under U(1) as $S = (\mathbb{C} < 1, e_{12}, e_{13} >)_{\mathbb{R}} \oplus^4 (\mathbb{C})_{\mathbb{R}}$, where U(1) acts on $\mathbb{C}$ with the fundamental representation. Thus there are six $U(1) \ltimes \mathbb{R}^8$-invariant spinors. In addition, the orbits of SU(2) in $\oplus^2(\mathbb{C}^2)_{\mathbb{R}}$ have stability subgroup $\{1\}$ in Spin(9,1). Thus there are no more invariant spinors. This concludes the cases with non-compact stability subgroups.

Next let us consider the descendants of G_2. The G_2 decomposition of S^+ is $S^+ = \mathbb{R} < 1 + e_{1234} > \oplus \mathbb{R} < e_{15} + e_{2345} > \oplus \Lambda^1(\mathbb{R}^7) \oplus \Lambda^1(\mathbb{R}^7)$. In addition G_2 has a single orbit in $\Lambda^1(\mathbb{R}^7)$ of co-dimension one which has stability subgroup SU(3). Moreover S^+ under SU(3) decomposes as $S^+ = (\mathbb{C} < 1, e_{15} >)_{\mathbb{R}} \oplus (\Lambda^2(\mathbb{C}^3))_{\mathbb{R}} \oplus (\Lambda^1(\mathbb{C}^3))_{\mathbb{R}}$, thus there are four SU(3)-invariant spinors. In either $(\Lambda^2(\mathbb{C}^3))_{\mathbb{R}}$ or $(\Lambda^1(\mathbb{C}^3))_{\mathbb{R}}$, SU(3) acts with stability subgroup SU(2). In addition, S^+ decomposes under SU(2) as $S^+ = (\mathbb{C} < 1, e_{15}, e_{12}, e_{25} >)_{\mathbb{R}} \oplus^2 (\mathbb{C}^2)_{\mathbb{R}}$.

[16] With $V_{\mathbb{R}}$ we denote the associated real representation of a complex representation, i.e. $(\mathbb{C} < 1 >)_{\mathbb{R}} = \mathbb{R} < (1 + e_{1234}, i(1 - e_{1234}) >$.

Thus there are eight SU(2)-invariant spinors. Moreover, the orbits of SU(2) in $\oplus^2(\mathbb{C}^2)_\mathbb{R}$ have stability subgroup $\{1\}$. So there are no other cases to investigate. This concludes the analysis.

C. SO(3) transformations

The first three SU(2)-invariant Killing spinors are given by $1 + e_{1234}$, $e_{15} + e_{2345}$ and $i(1 - e_{1234}) + e_{25} - e_{1345}$. Therefore the fourth Killing spinor is spanned by the following basis elements:

$$\begin{aligned}
&\lambda_1 = i(1 - e_{1234}) - e_{25} + e_{1345}\,, \quad \lambda_2 = i(e_{15} - e_{2345}\,, \quad \lambda_3 = i(e_{25} + e_{1345})\,,\\
&\lambda_4 = e_{12} - e_{34}\,, \quad \lambda_5 = i(e_{12} + e_{34})\,.
\end{aligned} \tag{C.1}$$

The action of the generators t_i of SO(3) on these is given by (omitting terms proportional to the first three Killing spinors, i.e. restricting to $\mathcal{P}/\mathcal{K}$ as discussed in the text)

$$\begin{aligned}
&t_1(\lambda_1) = -2\lambda_5\,, \quad t_2(\lambda_1) = 2\lambda_3\,, \quad t_3(\lambda_1) = 2(\lambda_2 + \lambda_4)\,,\\
&t_1(\lambda_2) = -2\lambda_3\,, \quad t_2(\lambda_2) = 0\,, \quad t_3(\lambda_2) = -\lambda_1\,,\\
&t_1(\lambda_3) = 2\lambda_2 - \lambda_4\,, \quad t_2(\lambda_3) = -\tfrac{1}{2}\lambda_1\,, \quad t_3(\lambda_3) = -\lambda_5\,,\\
&t_1(\lambda_4) = 0\,, \quad t_2(\lambda_4) = -2\lambda_5\,, \quad t_3(\lambda_4) = -\lambda_1\,,\\
&t_1(\lambda_5) = \tfrac{1}{2}\lambda_1\,, \quad t_2(\lambda_5) = -\lambda_2 + 2\lambda_4\,, \quad t_3(\lambda_5) = \lambda_3\,.
\end{aligned} \tag{C.2}$$

One can also see explicitly that the λ's constitute the symmetric traceless representation of SO(3). Define the matrix

$$M = \begin{pmatrix} 2\lambda_4 & \lambda_1 & 2\lambda_5 \\ \lambda_1 & -2\lambda_2 & -2\lambda_3 \\ 2\lambda_5 & -2\lambda_3 & 2\lambda_2 - 2\lambda_4 \end{pmatrix}. \tag{C.3}$$

The transformation (C.2) corresponds to

$$t_i(M) = M t_i - t_i M\,, \tag{C.4}$$

with the generators given by

$$t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{C.5}$$

From these formulae it is clear that while the first three Killing spinors transform in the fundamental representation of SO(3), the remaining five basis elements form the symmetric traceless representation.

D. Null spinor bilinears

The spinor bilinear vectors for the extra two basis elements e_{13} and e_{24} are given by

$$\kappa(e_{13}, e_{24}) = (e^0 - e^5)\,, \tag{D.1}$$

and other combinations vanishing. Similarly, the non-vanishing bilinear three-forms are given by

$$\begin{aligned}
\xi(1, e_{13}) &= -(e^0 - e^5) \wedge (e^2 + ie^7) \wedge (e^4 + ie^9)\,,\\
\xi(e_{1234}, e_{13}) &= -(e^0 - e^5) \wedge (e^1 - ie^6) \wedge (e^3 - ie^8)\,,\\
\xi(e_{12}, e_{13}) &= -(e^0 - e^5) \wedge (e^1 - ie^6) \wedge (e^4 + ie^9)\,,\\
\xi(e_{24}, e_{13}) &= -i(e^0 - e^5) \wedge (\omega_1 - \omega_2)\,,\\
\xi(e_{34}, e_{13}) &= -(e^0 - e^5) \wedge (e^2 + ie^7) \wedge (e^3 - ie^8)\,,\\
\xi(1, e_{24}) &= -(e^0 - e^5) \wedge (e^1 + ie^6) \wedge (e^3 + ie^8)\,,\\
\xi(e_{1234}, e_{24}) &= -(e^0 - e^5) \wedge (e^2 - ie^7) \wedge (e^4 - ie^9)\,,\\
\xi(e_{12}, e_{24}) &= (e^0 - e^5) \wedge (e^2 - ie^7) \wedge (e^3 + ie^8)\,,\\
\xi(e_{34}, e_{24}) &= (e^0 - e^5) \wedge (e^1 + ie^6) \wedge (e^4 - ie^9)\,.
\end{aligned} \tag{D.2}$$

Finally, the non-vanishing bilinear five-forms are

$$\begin{aligned}
\tau(1, e_{13}) &= i(e^0 - e^5) \wedge (e^2 + ie^7) \wedge (e^4 + ie^9) \wedge \omega_1\,,\\
\tau(e_{1234}, e_{13}) &= -i(e^0 - e^5) \wedge (e^1 - ie^6) \wedge (e^3 - ie^8) \wedge \omega_2\,,\\
\tau(e_{12}, e_{13}) &= -i(e^0 - e^5) \wedge (e^1 - ie^6) \wedge (e^4 + ie^9) \wedge (e^2 \wedge e^7 - e^3 \wedge e^8)\,,\\
\tau(e_{13}, e_{13}) &= (e^0 - e^5) \wedge (e^1 - ie^6) \wedge (e^2 + ie^7) \wedge (e^3 - ie^8) \wedge (e^4 + ie^9)\,,\\
\tau(e_{24}, e_{13}) &= -\tfrac{1}{2}(e^0 - e^5) \wedge (\omega_1 - \omega_2) \wedge (\omega_1 - \omega_2)\,,\\
\tau(e_{34}, e_{13}) &= i(e^0 - e^5) \wedge (e^2 + ie^7) \wedge (e^3 - ie^8) \wedge (e^1 \wedge e^6 - e^4 \wedge e^9)\,,\\
\tau(1, e_{24}) &= i(e^0 - e^5) \wedge (e^1 + ie^6) \wedge (e^3 + ie^8) \wedge \omega_2\,,\\
\tau(e_{1234}, e_{24}) &= -i(e^0 - e^5) \wedge (e^2 - ie^7) \wedge (e^4 - ie^9) \wedge \omega_1\,,\\
\tau(e_{12}, e_{24}) &= i(e^0 - e^5) \wedge (e^2 - ie^7) \wedge (e^3 + ie^8) \wedge (e^1 \wedge e^6 - e^4 \wedge e^9)\,,\\
\tau(e_{24}, e_{24}) &= (e^0 - e^5) \wedge (e^1 + ie^6) \wedge (e^2 - ie^7) \wedge (e^3 + ie^8) \wedge (e^4 - ie^9)\,,\\
\tau(e_{34}, e_{24}) &= -i(e^0 - e^5) \wedge (e^1 + ie^6) \wedge (e^4 - ie^9) \wedge (e^2 \wedge e^7 - e^3 \wedge e^8)\,.
\end{aligned} \tag{D.3}$$

Here we have used the following definitions

$$\omega_1 = e^1 \wedge e^6 + e^3 \wedge e^8\,, \quad \omega_2 = e^2 \wedge e^7 + e^4 \wedge e^9\,, \tag{D.4}$$

for the two-forms.

From these expressions one can derive the inner products for the null Majorana spinors, which are

$$\begin{aligned}
\epsilon_1 &= 1 + e_{1234}\,, & \epsilon_2 &= i(1 - e_{1234})\,,\\
\epsilon_3 &= e_{12} - e_{34}\,, & \epsilon_4 &= i(e_{12} + e_{34})\,,\\
\epsilon_5 &= e_{13} + e_{24}\,, & \epsilon_6 &= i(e_{13} - e_{24})\,.
\end{aligned} \tag{D.5}$$

In the Majorana basis of spinors, the bilinear vectors read

$$\begin{aligned}\kappa(\epsilon_5, \epsilon_5) &= 2(e^0 - e^5)\,,\\ \kappa(\epsilon_6, \epsilon_6) &= 2(e^0 - e^5)\,.\end{aligned} \tag{D.6}$$

The bilinear three-forms are given by

$$\begin{aligned}
\xi(\epsilon_1, \epsilon_5) &= -2(e^0 - e^5) \wedge (e^2 \wedge e^4 - e^7 \wedge e^9 + e^1 \wedge e^3 - e^6 \wedge e^8)\,,\\
\xi(\epsilon_1, \epsilon_6) &= -2(e^0 - e^5) \wedge (-e^2 \wedge e^9 + e^4 \wedge e^7 + e^1 \wedge e^8 - e^3 \wedge e^6)\,,\\
\xi(\epsilon_2, \epsilon_5) &= -2(e^0 - e^5) \wedge (-e^2 \wedge e^9 + e^4 \wedge e^7 - e^1 \wedge e^8 + e^3 \wedge e^6)\,,\\
\xi(\epsilon_2, \epsilon_6) &= 2(e^0 - e^5) \wedge (e^2 \wedge e^4 - e^7 \wedge e^9 - e^1 \wedge e^3 + e^6 \wedge e^8)\,,\\
\xi(\epsilon_3, \epsilon_5) &= -2(e^0 - e^5) \wedge (e^1 \wedge e^4 + e^6 \wedge e^9 - e^2 \wedge e^3 - e^7 \wedge e^8)\,,\\
\xi(\epsilon_3, \epsilon_6) &= -2(e^0 - e^5) \wedge (-e^1 \wedge e^9 - e^4 \wedge e^6 - e^2 \wedge e^8 - e^3 \wedge e^7)\,,\\
\xi(\epsilon_4, \epsilon_5) &= -2(e^0 - e^5) \wedge (-e^1 \wedge e^9 - e^4 \wedge e^6 + e^2 \wedge e^8 + e^3 \wedge e^7)\,,\\
\xi(\epsilon_4, \epsilon_6) &= 2(e^0 - e^5) \wedge (e^1 \wedge e^4 + e^6 \wedge e^9 + e^2 \wedge e^3 + e^7 \wedge e^8)\,,\\
\xi(\epsilon_5, \epsilon_6) &= 2(e^0 - e^5) \wedge (\omega_1 - \omega_2)\,.
\end{aligned} \tag{D.7}$$

Finally, the five-forms are

$$\begin{aligned}
\tau(\epsilon_i, \epsilon_5) &= -\xi(\epsilon_i, \epsilon_6) \wedge (\omega_1 - \omega_2)\,, \quad i = 1, \ldots, 4,\\
\tau(\epsilon_i, \epsilon_6) &= \xi(\epsilon_i, \epsilon_5) \wedge (\omega_1 - \omega_2)\,, \quad i = 1, \ldots, 4,\\
\tau(\epsilon_5, \epsilon_5) &= -(e^0 - e^5) \wedge (\omega_1 - \omega_2) \wedge (\omega_1 - \omega_2) + 2(e^0 - e^5) \wedge \mathrm{Re}(\chi)\,,\\
\tau(\epsilon_5, \epsilon_6) &= -2(e^0 - e^5) \wedge \mathrm{Im}(\chi)\,,\\
\tau(\epsilon_6, \epsilon_6) &= -(e^0 - e^5) \wedge (\omega_1 - \omega_2) \wedge (\omega_1 - \omega_2) - 2(e^0 - e^5) \wedge \mathrm{Re}(\chi)\,,
\end{aligned} \tag{D.8}$$

where we have used

$$\chi = (e^1 - ie^6) \wedge (e^2 + ie^7) \wedge (e^3 - ie^8) \wedge (e^4 + ie^9)\,. \tag{D.9}$$

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